Control of Linear Parameter Varying Systems

by

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Dedicated to my wife
Juan Huang
and our daughter
Vanessa Shirley Wu,
with all my love.

Contents

Notations

Mathematical Symbols

$\mathbf{R}$	set of real numbers
$\mathbf{R}_+$	set of non-negative real numbers
$\mathbf{R}^n$	set of n-dimensional real vectors
$\mathbf{R}^{n\times m}$	set of n by m matrices with elements in $\mathbf{R}$
$\mathcal{S}^{n\times n}$	set of symmetric matrices in $\mathbf{R}^{n\times n}$
$\mathcal{S}_+^{n\times n}$	set of positive definite matrices in $\mathbf{R}^{n\times n}$
M^T	the transpose of the matrix M
M^{-1}	the inverse of the invertible matrix M
$tr(M)$	the trace of the square matrix M
$diag(a_1, a_2, \cdots, a_n)$	the n by n diagonal matrix with elements $a_1, a_2, \cdots, a_n$ on the diagonal line
$\mathcal{E}(X)$	the expected value of the random variable X
I_n	the n-dimensional identity
$0_{n\times m}$	the zero element of $\mathbf{R}^{n\times m}$

For $M \in \mathcal{S}^{n\times n}$

$\lambda_{\max}(M)$	the maximum eigenvalue of M
$M > 0$ ($M \geq 0$)	M is positive definite (positive semi-definite)
$M < 0$ ($M \leq 0$)	M is negative definite (negative semi-definite)
$M^{\frac{1}{2}}$	Hermitian square root of M

Additional Notations

$\|\cdot\|$	vector norm
$\|\cdot\|_2$	$\mathbf{L}_2$-norm
$\|\cdot\|_F$	Frobenius norm
$\|\cdot\|_{i,2}$	induced $\mathbf{L}_2$-norm
ρ	plant parameter vector
$\mathcal{P}$	compact parameter set
$\mathcal{F}_{\mathcal{P}}$	parameter variation set, Definition 1.1.1
$\mathcal{F}_{\mathcal{P}}^{\nu}$	parameter ν-variation set, Definition 3.2.1
$G_{\mathcal{F}_{\mathcal{P}}^{\nu}}$	set of causal linear operators defined on $\mathcal{F}_{\mathcal{P}}^{\nu}$, Definition 3.2.2
$\mathcal{C}^0(U,V)$	set of continuous functions from U to V
$\mathcal{C}^1(U,V)$	set of continuously differentiable functions from U to V
$closure(U)$	closure of the set U
$\mathcal{F}_{\ell}(\cdot,\cdot)$	linear fractional transformation
$:=$	"is defined to be"
$\blacksquare$	end of proof
§	section

Acronyms

FD	finite dimensional
LFT	linear fractional transformation
LQG	linear quadratic Gaussian
AMI	affine matrix inequality
LMI	linear matrix inequality
LTI	linear time-invariant
LTV	linear time-varying
LPV	linear parameter varying
SQLF	single quadratic Lyapunov function
PDLF	parameter dependent Lyapunov function
QS	quadratically stabilizable
QD	quadratically detectable
PDS	parametrically-dependent stabilizable
PDD	parametrically-dependent detectable

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Part I

LQG Control of LPV Systems

Chapter 1

Analysis of Linear Parameter Varying Systems

In this chapter, we first introduce the concept of LPV systems and quadratic stability, then define LQG performance for LPV systems and derive performance bounds through two analysis results.

In §1.1, we precisely define the class of systems we focus on in this thesis–linear parameter varying (LPV) systems. In §1.2, we state the definition of quadratic stability and its relation with other notions of stability, and derive some equivalent conditions for quadratic stability. In §1.3, we talk about quadratic stabilizability and detectability. In §1.4, we define LQG performance for quadratically stable LPV systems and give two performance bounds. In §1.5, we discuss the computational aspects of these analysis results. In §1.6, we compare our performance bounds with previous bounds in [PetH], [BerH4] .

1.1 LPV Systems

Before introducing the LPV systems, we need to define the set of all admissible parameter trajectories. The restrictions in the definition guarantee the existence and uniqueness of the solution to the differential equation governing an LPV system.

Definition 1.1.1 Parameter Variation Set
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, the parameter variation set $\mathcal{F}_{\mathcal{P}}$ denotes the set of all piecewise continuous functions mapping $\mathbf{R}^+$ (time) into $\mathcal{P}$ with a finite number of discontinuities in any interval.

Remark 1.1.1 *In the remainder of this thesis, the notation $\rho \in \mathcal{F}_{\mathcal{P}}$ denotes a time-varying trajectory in the parameter variation set, while $\rho \in \mathcal{P}$ denotes a vector in a compact subset of $\mathbf{R}^s$.*

Now we define LPV systems which will be the focus of this thesis.

Definition 1.1.2 Linear Parameter Varying (LPV) System
Assume that the following are given:

- *a compact set $\mathcal{P} \subset \mathbf{R}^s$,*
- *a function $A \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n \times n})$,*
- *a function $B \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n \times n_d})$,*
- *a function $C \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n_e \times n})$, and*
- *a function $D \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n_e \times n_d})$.*

An n-th order linear parameter varying (LPV) system is the one whose dynamics evolve as

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & D(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, \tag{1.1.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$, $x(t), \dot{x}(t) \in \mathbf{R}^n$, $d(t) \in \mathbf{R}^{n_d}$ and $e(t) \in \mathbf{R}^{n_e}$.

Remark 1.1.2 *As the matrix functions A, B, C and D are continuous functions of parameter ρ, they are, in fact, norm-bounded on the compact set $\mathcal{P}$.*

The following definition summarizes some notation that will be used for the rest of the thesis.

Definition 1.1.3 *For any $\rho \in \mathcal{F}_{\mathcal{P}}$,*

- *the linear time-varying system described by equation (1.1.1), is denoted Σ_ρ,*
- *$\Phi_\rho(t, t_0)$ is the state-transition matrix of Σ_ρ.*

The notation $\Sigma_{\mathcal{P}} := \{\Sigma_\rho : \rho \in \mathcal{F}_{\mathcal{P}}\}$ represents the LPV system in Definition 1.1.2, we will sometimes use $\Sigma(\mathcal{P}, A, B, C, D)$ to illustrate the state-space date clearly.

1.2 Quadratic Stability of LPV Systems

In this section we state the concept of quadratic stability for LPV systems, and derive equivalent conditions. Quadratic stability [Bar] is a strong notion of robust stability in the sense that it holds for arbitrarily fast variations in the parameter trajectory ρ, and its definition involves a single quadratic Lyapunov function.

Definition 1.2.1 Quadratic Stability
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, and a function $A \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n\times n})$, the function A is quadratically stable over $\mathcal{P}$ if there exists a matrix $P \in \mathcal{S}_+^{n\times n}$, such that for all $\rho \in \mathcal{P}$

$$A^T(\rho)P + PA(\rho) < 0. \tag{1.2.1}$$

Remark 1.2.1 *Since A is a continuous function of parameter $\rho \in \mathcal{P}$, and $\mathcal{P}$ is compact, it is clear that condition (1.2.1) implies that the left hand side is uniformly negative definite. That is, there exists a scalar $\delta > 0$, such that for all $\rho \in \mathcal{P}$, $A^T(\rho)P + PA(\rho) \leq -\delta I_n$.*

Next we formulate different necessary and sufficient conditions for quadratic stability.

Theorem 1.2.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, and an integer $m > 0$, then the following conditions are equivalent:*

(1) function $A(\rho)$ is quadratically stable over $\mathcal{P}$,

(2) for any $C \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{m\times n})$, there exists a matrix $X \in \mathcal{S}_+^{n\times n}$, such that for all $\rho \in \mathcal{P}$

$$A(\rho)X + XA^T(\rho) + XC^T(\rho)C(\rho)X < 0, \tag{1.2.2}$$

(3) for any $B \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n\times m})$, there exists a matrix $Y \in \mathcal{S}_+^{n\times n}$, such that for all $\rho \in \mathcal{P}$

$$A^T(\rho)Y + YA(\rho) + YB(\rho)B^T(\rho)Y < 0. \tag{1.2.3}$$

Proof: (1) $\Rightarrow$ (2) From Remark 1.2.1 and quadratic stability assumption of function $A(\rho)$, there exists a matrix $X \in \mathcal{S}_+^{n\times n}$, such that for all $\rho \in \mathcal{P}$

$$X^{-1}A(\rho) + A^T(\rho)X^{-1} \leq -\delta I. \tag{1.2.4}$$

Note $C(\rho)$ is continuous function on $\mathcal{P}$ thus is bounded, so there exists a scalar $\gamma > 0$ such that for all $\rho \in \mathcal{P}$

$$C^T(\rho)C(\rho) < \gamma\delta I. \tag{1.2.5}$$

Multiplying equation (1.2.4) by γ and adding to equation (1.2.5), we get

$$\gamma X^{-1}A(\rho) + A^T(\rho)\gamma X^{-1} + C^T(\rho)C(\rho) < 0$$

for all $\rho \in \mathcal{P}$. Redefining $\tilde{X} = \frac{1}{\gamma}X$, gives condition (1.2.2).
(2) $\Rightarrow$ (1) If the condition (1.2.2) holds, then for all $\rho \in \mathcal{P}$

$$A(\rho)X + XA^T(\rho) < -XC^T(\rho)C(\rho)X < 0,$$

that is

$$X^{-1}A(\rho) + A^T(\rho)X^{-1} < 0$$

for all $\rho \in \mathcal{P}$. So function $A(\rho)$ is quadratically stable by Definition 1.2.1. Similarly, we can prove that (1) $\Leftrightarrow$ (3). ∎

The next notion is natural after the definition of quadratically stable function.

Definition 1.2.2 Quadratically Stable LPV System
For LPV system $\Sigma_{\mathcal{P}}$ in Definition 1.1.2, if A is quadratically stable over $\mathcal{P}$, then $\Sigma_{\mathcal{P}}$ is a quadratically stable LPV system.

Also we may define exponential stability for LPV systems as follows.

Definition 1.2.3 Exponential Stability
The LPV system in Definition 1.1.2 is exponentially stable if there exist some constants $M, \alpha > 0$, such that for all $\rho \in \mathcal{F}_{\mathcal{P}}$ and all $t \geq \tau$

$$\|\Phi_\rho(t,\tau)\| \leq Me^{[\alpha(t-\tau)]}.$$

It is well known that for LTI systems, quadratic stability is equivalent to exponential stability [Vid]. But for LPV systems, these two concepts are not the same any more. Actually the quadratic stability implies exponential stability, which is shown in the following lemma.

Lemma 1.2.1 *Given a compact set $\mathcal{P}$, and a quadratically stable LPV system*

$$\dot{x}(t) = A(\rho(t))x(t), \tag{1.2.6}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. There exist constant scalars $\gamma_1, \gamma_2 > 0$ such that the state-transition matrix $\Phi_\rho(t, t_0)$, which characterizes all solutions to equation (1.2.6), satisfies

$$\|\Phi_\rho(t, t_0)\| \leq \gamma_1 \, e^{[-\gamma_2(t-t_0)]}$$

for all $\rho \in \mathcal{F}_{\mathcal{P}}$.

Proof: Using the fact that there exists a positive definite matrix P, such that

$$\begin{aligned} A(\rho)P + PA(\rho) &\leq -\delta I \\ \lambda_{min} I \leq P &\leq \lambda_{max} I \end{aligned}$$

where δ, $\lambda_{min}, \lambda_{max}$ are some positive numbers, the proof follows with $\gamma_1 = (\lambda_{max}/\lambda_{min})^{\frac{1}{2}}$ and $\gamma_2 = (\delta/(2\lambda_{max}))$. ∎

1.3 Quadratic Stabilizability and Detectability

After the notion of quadratic stability, we will introduce quadratic stabilizability and detectability concepts in this section, and formulate equivalent conditions for them. These conditions will be used later in Chapter 3.

1.3.1 Quadratic Stabilizability

The notation of quadratic stabilizability is given by

Definition 1.3.1 Quadratic Stabilizability
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, the pair of matrix functions (A, B_2) is quadratically stabilizable (QS) over $\mathcal{P}$ if there exist a matrix $P_F \in \mathcal{S}_+^{n\times n}$, and a function $F \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n_u \times n})$ such that

$$[A(\rho) + B_2(\rho)F(\rho)]^T P_F + P_F [A(\rho) + B_2(\rho)F(\rho)] < 0$$

for all $\rho \in \mathcal{P}$. Such a function F is called a quadratically stabilizing state-feedback gain for the pair (A, B_2) over $\mathcal{P}$.

The following theorem states the equivalent conditions for quadratic stabilizability.

Theorem 1.3.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$. For any continuous functions $(C_{11}, C_{12}) : \mathbf{R}^s \to (\mathbf{R}^{n_{e1} \times n}, \mathbf{R}^{n_u \times n})$, the following conditions are equivalent:*

(1) the pair (A, B_2) is QS over $\mathcal{P}$,

(2) there exist a matrix $X \in \mathcal{S}_+^{n \times n}$ and a function $F \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n_u \times n})$, such that for all $\rho \in \mathcal{P}$

$$A_F(\rho)X + XA_F^T(\rho) + XC_F^T(\rho)C_F(\rho)X < 0 \tag{1.3.1}$$

where $A_F(\rho) := A(\rho) + B_2(\rho)F(\rho)$, $C_F^T(\rho) := \begin{bmatrix} C_{11}^T(\rho) & C_{12}^T(\rho) + F^T(\rho) \end{bmatrix}$,

(3) there exists a matrix $X \in \mathcal{S}_+^{n \times n}$, such that for all $\rho \in \mathcal{P}$

$$\begin{bmatrix} \hat{A}(\rho)X + X\hat{A}^T(\rho) - B_2(\rho)B_2^T(\rho) & XC_{11}^T(\rho) \\ C_{11}(\rho)X & -I_{n_{e1}} \end{bmatrix} < 0 \tag{1.3.2}$$

where $\hat{A}(\rho) := A(\rho) - B_2(\rho)C_{12}(\rho)$.

Proof: It is straight forward to show that (1) $\Leftrightarrow$ (2) using condition (1.2.2) in Theorem 1.2.1.

(2) $\Rightarrow$ (3) Condition (1.3.1) is equivalent to

$$X^{-1}A_F(\rho) + A_F^T(\rho)X^{-1} + C_F^T(\rho)C_F(\rho) < 0 \tag{1.3.3}$$

for all $\rho \in \mathcal{P}$. By Schur complement, equation (1.3.3) can be rewritten as

$$\begin{bmatrix} X^{-1}A_F(\rho) + A_F^T(\rho)X^{-1} & C_F^T(\rho) \\ C_F(\rho) & -I_{n_e} \end{bmatrix} < 0 \tag{1.3.4}$$

for all $\rho \in \mathcal{P}$. Define matrix functions $R(\rho)$, $U(\rho)$ and $V(\rho)$ as follows:

$$R(\rho) \;:=\; \begin{bmatrix} X^{-1}A(\rho) + A^T(\rho)X^{-1} & C_{11}^T(\rho) & C_{12}^T(\rho) \\ C_{11}(\rho) & -I_{n_{e1}} & 0 \\ C_{12}(\rho) & 0 & -I_{n_{e2}} \end{bmatrix},$$

$$U(\rho) \;:=\; \begin{bmatrix} X^{-1}B_2(\rho) \\ 0 \\ I_{n_{e2}} \end{bmatrix}, \qquad V(\rho) := [I_n \;\; 0 \;\; 0].$$

Now we can rewrite equation (1.3.4) as

$$G(\rho) := R(\rho) + U(\rho)F(\rho)V(\rho) + V^T(\rho)F^T(\rho)U^T(\rho) < 0$$

for all $\rho \in \mathcal{P}$. The orthonormal bases of $U(\rho)$, $V(\rho)$ are

$$U_\perp(\rho) = \begin{bmatrix} X & 0 \\ 0 & I_{n_{e1}} \\ -B_2^T(\rho) & 0 \end{bmatrix}, \quad V_\perp(\rho) = \begin{bmatrix} 0 & 0 \\ I_{n_{e1}} & 0 \\ 0 & I_{n_{e2}} \end{bmatrix}.$$

So if $G(\rho) < 0$ for all $\rho \in \mathcal{P}$, then

$$U_\perp^T(\rho)G(\rho)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp(\rho)G(\rho)V_\perp^T(\rho) < 0$$

hold for all $\rho \in \mathcal{P}$. This implies that

$$U_\perp^T(\rho)R(\rho)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp(\rho)R(\rho)V_\perp^T(\rho) < 0$$

for all $\rho \in \mathcal{P}$. Note that $V_\perp(\rho)R(\rho)V_\perp^T(\rho) < 0$ yields no useful information. But by simple algebra, we can verify that $U_\perp^T(\rho)R(\rho)U_\perp(\rho) < 0$ is identical to condition (1.3.2).
(2) $\Leftarrow$ (3) By Schur complement, condition (1.3.2) is equivalent to

$$\hat{A}(\rho)X + X\hat{A}^T(\rho) - B_2(\rho)B_2^T(\rho) + XC_{11}^T(\rho)C_{11}(\rho)X < 0, \quad \forall\, \rho \in \mathcal{P}$$

which is equivalent to

$$\begin{aligned} &\left[A - B_2\left(B_2^T X^{-1} + C_{12}\right)\right] X + X\left[A - B_2\left(B_2^T X^{-1} + C_{12}\right)\right]^T \\ &\quad + X\left[C_{11}^T \;\; C_{12}^T - \left(X^{-1}B_2 + C_{12}^T\right)\right] \begin{bmatrix} C_{11} \\ C_{12} - \left(B_2^T X^{-1} + C_{12}\right) \end{bmatrix} X < 0. \end{aligned} \tag{1.3.5}$$

This results in a natural choice of state-feedback gain $F(\rho) = -\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]$, so that equation (1.3.5) can be rewritten as

$$A_F(\rho)X + XA_F^T(\rho) + XC_F^T(\rho)C_F(\rho)X < 0, \quad \forall\, \rho \in \mathcal{P}.$$

Clearly, the pair (A, B_2) is QS with quadratically stabilizing gain F given above. ∎

By slight abuse of quadratic stabilizability concept, we can formulate equivalent conditions for quadratic stabilization using static state-feedback as follows.

Theorem 1.3.2 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$. for any continuous functions $(C_1, D_{12}) : \mathbf{R}^s \to (\mathbf{R}^{n_{e1}\times n}, \mathbf{R}^{n_{e1}\times n_u})$, the following conditions are equivalent:*

(1) the pair (A, B_2) is QS by static state-feedback gain F,

(2) There exist matrices $X \in \mathcal{S}_+^{n\times n}$ and $R \in \mathbf{R}^{n_u\times n}$ such that for all $\rho \in \mathcal{P}$

$$\begin{bmatrix} A(\rho)X + XA^T(\rho) + B_2(\rho)R + R^T B_2^T(\rho) & XC_1^T(\rho) + R^T D_{12}^T(\rho) \\ C_1(\rho)X + D_{12}(\rho)R & -I \end{bmatrix} < 0, \tag{1.3.6}$$

(3) There exist matrices $Y \in \mathcal{S}_+^{n\times n}$ and $S \in \mathbf{R}^{n_u\times n}$ such that for all $\rho \in \mathcal{P}$

$$A(\rho)Y + YA^T(\rho) + B_2(\rho)S + S^T B_2^T(\rho) + B_1(\rho)B_1^T(\rho) < 0. \tag{1.3.7}$$

Proof: (1) $\Rightarrow$ (2) Note that the pair (A, B_2) is QS by constant F means $A_F(\rho) := A(\rho) + B_2(\rho)F$ is quadratically stable. So there exists a matrix $X \in \mathcal{S}_+^{n\times n}$, such that

$$A_F(\rho)X + XA_F^T(\rho) + XC_F^T(\rho)C_F(\rho)X < 0$$

for any $C_F(\rho) := C_1(\rho) + D_{12}(\rho)F$ and all $\rho \in \mathcal{P}$ (condition (1.2.2) in Theorem 1.2.1). Define $R = FX$, the above inequality can be written as

$$A(\rho)X + XA^T(\rho) + B_2(\rho)R + R^T B_2^T(\rho) + \left[XC_1^T(\rho) + R^T D_{12}^T(\rho)\right]\left[C_1(\rho)X + D_{12}(\rho)R\right] < 0 \tag{1.3.8}$$

for all $\rho \in \mathcal{P}$. By Schur complement arguments, it is easy to see that equation (1.3.8) is equivalent to the equation (1.3.6).

(1) $\Leftarrow$ (2) The argument given above is reversed.

(1) $\Leftrightarrow$ (3) It can be shown similarly by condition (1.2.3) in Theorem 1.2.1. ∎

1.3.2 Quadratic Detectability

First note, quadratic detectability is the dual of quadratic stabilizability.

Definition 1.3.2 Quadratic Detectability

Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, the pair (A, C_2) is quadratically detectable (QD) over $\mathcal{P}$ if there exist a matrix $P_L \in \mathcal{S}_+^{n\times n}$ and a function $L \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n\times n_y})$, such that

$$P_L\left[A(\rho) + L(\rho)C_2(\rho)\right] + \left[A(\rho) + L(\rho)C_2(\rho)\right]^T P_L < 0$$

for all $\rho \in \mathcal{P}$.

Similar to Theorem 1.3.1, we have the following theorem to give the equivalent conditions for quadratic detectability.

Theorem 1.3.3 *Given a compact set* $\mathcal{P} \subset \mathbf{R}^s$. *For any continuous functions* $(B_{11}, B_{12}) : \mathbf{R}^s \to (\mathbf{R}^{n \times n_{d1}}, \mathbf{R}^{n \times n_y})$, *the following conditions are equivalent:*

(1) the pair (A, C_2) *is QD over* $\mathcal{P}$,

(2) there exist a matrix $Y \in \mathcal{S}_+^{n \times n}$ *and a function* $L \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n_y \times n})$, *such that for all* $\rho \in \mathcal{P}$

$$A_L^T(\rho)Y + YA_L(\rho) + YB_L(\rho)B_L^T(\rho)Y < 0$$

where $A_L(\rho) := A(\rho) + L(\rho)C_2(\rho)$, $B_L(\rho) := [B_{11}(\rho) \quad B_{12}(\rho) + L(\rho)]$,

(3) there exists a matrix $Y \in \mathcal{S}_+^{n \times n}$, *such that for all* $\rho \in \mathcal{P}$

$$\begin{bmatrix} \tilde{A}^T(\rho)Y + Y\tilde{A}(\rho) - C_2^T(\rho)C_2(\rho) & YB_{11}(\rho) \\ B_{11}^T(\rho)Y & -I \end{bmatrix} < 0 \qquad (1.3.9)$$

where $\tilde{A}(\rho) := A(\rho) - B_{12}(\rho)C_2(\rho)$.

Proof: The equivalence of (A, B_2) QS to condition (2) is clear from condition (1.2.3) in Theorem 1.2.1. The proof for (2) $\Leftrightarrow$ (3) is similar to the one for Theorem 1.3.1 with

$$L(\rho) := -\left[Y^{-1}C_2^T(\rho) + B_{12}(\rho)\right].$$

∎

1.4 LQG Performance and Analysis for LPV Systems

In this section we define an LQG performance measure for LPV systems. Based on the equivalent conditions for quadratic stability in Theorem 1.2.1, two LQG performance bounds for LPV systems are formulated in Theorem 1.4.1 and Theorem 1.4.2.

1.4.1 LQG Performance Measure

Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, and the LPV system in Definition 1.1.2 with its "D" term equal to zero, that is,

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & 0_{n_e \times n_d} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, \tag{1.4.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. The initial state $x(0)$ is a stochastic variable and independent of the white noise $d(t)$, with

$$\mathcal{E}\{x(0)\} =: \bar{x}_0, \tag{1.4.2.a}$$

$$\mathcal{E}\left\{(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)^T\right\} =: Q_0, \tag{1.4.2.b}$$

$$\mathcal{E}\left\{d(t_1)d^T(t_2)\right\} =: V\,\delta(t_1 - t_2), \tag{1.4.2.c}$$

where $Q_0 > 0$, $V \geq 0$, are given.

We are concerned with the "worst-case" measure of LQG performance over all admissible trajectories belonging to $\mathcal{F}_{\mathcal{P}}$.

Definition 1.4.1 LQG Performance of LPV Systems
Given a compact set $\mathcal{P}$, and a quadratically stable LPV system in (1.4.1) and (1.4.2.), define the LQG performance over finite interval $[0, T]$ as

$$\sigma_T := \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \mathcal{E}\left\{\frac{1}{T}\int_0^T e^T(t)e(t)\,dt\right\}.$$

Furthermore, over the infinite horizon $[0, \infty)$ let $\sigma_\infty := \lim_{T \to \infty} \sigma_T$.

Lemma 1.4.1 *Given a compact set $\mathcal{P}$, and a quadratically stable LPV system in (1.4.1) and (1.4.2.), then there exists a finite number $M > 0$, such that*

$$\sigma_T \leq M < \infty$$

for all $T \geq 0$. Obviously, we have $\sigma_\infty \leq M$.

Proof: For any $\rho \in \mathcal{F}_{\mathcal{P}}$, the output of LPV system is given by

$$e(t) = C(\rho(t))\Phi_\rho(t,0)x(0) + \int_0^t C(\rho(t))\Phi_\rho(t,\tau)B(\rho(\tau))d(\tau)d\tau.$$

As functions B and C are continuous, they are bounded on the compact set $\mathcal{P}$ and there exist finite scalars $k_B, k_C > 0$, such that $\|B(\rho)\| < k_B$ and $\|C(\rho)\| < k_C$ for all $\rho \in \mathcal{P}$. Note that $x(0)$ is uncorrelated with $d(t)$, then

$$\begin{aligned}
\mathcal{E}\left\{\|e(t)\|^2\right\} &= \mathcal{E}\left\{\left\|C(\rho(t))\Phi_\rho(t,0)x(0) + \int_0^t C(\rho(t))\Phi_\rho(t,\tau)B(\rho(\tau))d(\tau)d\tau\right\|^2\right\} \\
&\leq \mathcal{E}\left\{\|C(\rho(t))\Phi_\rho(t,0)x(0)\|^2\right\} + \mathcal{E}\left\{\left\|\int_0^t C(\rho(t))\Phi_\rho(t,\tau)B(\rho(\tau))d(\tau)d\tau\right\|^2\right\} \\
&\leq k_C^2\|\Phi_\rho(t,0)\|^2\, tr\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + k_B^2 k_C^2\, tr(V)\int_0^t \|\Phi_\rho(t,\tau)\|^2 d\tau
\end{aligned}$$

From Lemma 1.2.1 and assumption of quadratically stable LPV system, there exist $\gamma_1, \gamma_2 > 0$, such that $\|\Phi_\rho(t,\tau)\| \leq \gamma_1 e^{[-\gamma_2(t-\tau)]}$. Integrating both sides from $\tau = 0$ to $\tau = t$, we get

$$\int_0^t \|\Phi_\rho(t,\tau)\|^2 d\tau \leq \frac{\gamma_1^2}{2\gamma_2},$$

and $\|\Phi_\rho(t,0)\|^2 \leq \gamma_1^2$ obviously. This gives

$$\mathcal{E}\left\{\|e(t)\|^2\right\} \leq k_C^2\gamma_1^2\, tr\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + k_B^2 k_C^2\, tr(V)\frac{\gamma_1^2}{2\gamma_2}.$$

Defining $M := k_C^2\gamma_1^2\, tr\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + k_B^2 k_C^2\, tr(V)\frac{\gamma_1^2}{2\gamma_2}$, we have

$$\sigma_T = \sup_{\rho\in\mathcal{P}} \mathcal{E}\left\{\frac{1}{T}\int_0^T e^T(t)e(t)\, dt\right\} \leq M < \infty.$$

The claim is also true for $T \to \infty$ and it can be shown by taking limit on both sides of above inequality. ∎

Lemma 1.4.1 suggests that it is meaningful to bound LQG performance σ_T (or σ_∞) for LPV system if it is quadratically stable. Later on, we will concentrate on the performance measure σ_∞.

1.4.2 Analysis of LPV Systems with LQG Performance

Now we are in the position to study analysis results for LPV systems, which are given in the unified framework of quadratic stability and LQG performance bounds.

Theorem 1.4.1 *Given a compact set $\mathcal{P}$ and a quadratically stable LPV system in (1.4.1) and (1.4.2.), define*

$$\mathcal{X} := \left\{X \in \mathcal{S}_+^{n\times n} :\ \max_{\rho\in\mathcal{P}} \lambda_{max}\left[A(\rho)X + XA^T(\rho) + XC^T(\rho)C(\rho)X\right] < 0\right\},$$

then

$$\sigma_\infty \leq \inf_{X\in\mathcal{X}} \max_{\rho\in\mathcal{P}} tr\left[X^{-1}B(\rho)VB^T(\rho)\right] =: \alpha$$

Proof: Since the system is quadratically stable, the set $\mathcal{X}$ is non-empty by condition (1.2.2) in Theorem 1.2.1. For fixed $X \in \mathcal{X}$ and any trajectory $\rho \in \mathcal{F}_\mathcal{P}$ on $[0, T]$, there exists a time-varying matrix function $\Delta(t) > 0$ such that

$$\frac{dX^{-1}}{dt} = 0 = X^{-1}A(\rho(t)) + A^T(\rho(t))X^{-1} + C^T(\rho(t))C(\rho(t)) + \Delta(t).$$

From classical optimal control theory ([KwaS, Theorem 1.54]), we have

$$\begin{aligned}
\sigma_T &= \sup_{\rho\in\mathcal{F}_\mathcal{P}} \mathcal{E}\left\{\frac{1}{T}\int_0^T x^T(t)C^T(\rho(t))C(\rho(t))x(t)\,dt\right\} \\
&\leq \sup_{\rho\in\mathcal{F}_\mathcal{P}} \frac{1}{T}\,\mathcal{E}\left\{\int_0^T x^T(t)\left[C^T(\rho(t))C(\rho(t)) + \Delta(t)\right]x(t)\,dt + x^T(T)X^{-1}x(T)\right\} \\
&\leq \sup_{\rho\in\mathcal{F}_\mathcal{P}} tr\left\{X^{-1}\left[\frac{1}{T}\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + \int_0^T B(\rho(t))VB^T(\rho(t))\,dt\right]\right\} \\
&\leq \inf_{X\in\mathcal{X}} \max_{\rho\in\mathcal{P}} tr\left\{X^{-1}\left[\frac{1}{T}\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + B(\rho)VB^T(\rho)\right]\right\}.
\end{aligned}$$

Taking limit $T \to \infty$ on both sides of above inequality leads to

$$\sigma_\infty \leq \max_{\rho\in\mathcal{P}} tr\left[X^{-1}B(\rho)VB^T(\rho)\right]. \tag{1.4.3}$$

Note the inequality in equation (1.4.3) holds for any $X \in \mathcal{X}$, so that

$$\sigma_\infty \leq \inf_{X\in\mathcal{X}} \max_{\rho\in\mathcal{P}} tr\left[X^{-1}B(\rho)VB^T(\rho)\right].$$

∎

The second analysis test is given in dual form of Theorem 1.4.1.

Theorem 1.4.2 *Given a compact set $\mathcal{P}$, and a quadratically stable LPV system in (1.4.1) and (1.4.2.). Define*

$$\mathcal{Y} := \left\{Y \in \mathcal{S}_+^{n\times n} :\ \max_{\rho\in\mathcal{P}} \lambda_{max}\left[YA(\rho) + A^T(\rho)Y + YB(\rho)VB^T(\rho)Y\right] < 0\right\},$$

then

$$\sigma_\infty \leq \inf_{Y<Q_0^{-1},\ Y\in\mathcal{Y}} \max_{\rho\in\mathcal{P}} tr\left[Y^{-1}C^T(\rho)C(\rho)\right] =: \beta.$$

Proof: Note the constraint in $\mathcal{Y}$ is the slight modifications of condition (1.2.3). By Theorem 1.2.1, the set $\mathcal{Y}$ is non-empty because the LPV system is quadratically stable. For a specific time-varying trajectory $\rho \in \mathcal{F}_{\mathcal{P}}$ on $[0, T]$, let $Q(t)$ be the matrix satisfying $Q(0) = Q_0$ and

$$\frac{d}{dt}Q(t) = A(\rho(t))Q(t) + Q(t)A^T(\rho(t)) + B(\rho(t))VB^T(\rho(t)). \tag{1.4.4}$$

then

$$Q(t) = \Phi_\rho(t,0)Q_0\Phi_\rho^T(t,0) + \int_0^t \Phi_\rho(t,\tau)B(\rho(\tau))VB^T(\rho(\tau))\Phi_\rho^T(t,\tau)\, d\tau.$$

Given $Y \in \mathcal{Y}$ with $Y < Q_0^{-1}$, there exists a time-varying matrix $\Delta(t) > 0$ such that

$$\frac{dY^{-1}}{dt} = 0 = A(\rho(t))Y^{-1} + Y^{-1}A^T(\rho(t)) + B(\rho(t))VB^T(\rho(t)) + \Delta(t). \tag{1.4.5}$$

Subtracting equation (1.4.5) from (1.4.4), we get

$$\frac{d}{dt}\left(Y^{-1} - Q(t)\right) = A(\rho(t))\left(Y^{-1} - Q(t)\right) + \left(Y^{-1} - Q(t)\right)A^T(\rho(t)) + \Delta(t).$$

This yields

$$Y^{-1} - Q(t) = \Phi_\rho(t,0)\left(Y^{-1} - Q_0\right)\Phi_\rho^T(t,0) + \int_0^t \Phi_\rho(t,\tau)\Delta(\tau)\Phi_\rho^T(t,\tau)\, d\tau \geq 0.$$

As $Y < Q_0^{-1}$, it follows $Y^{-1} \geq Q(t)$ for all $t \in [0, T]$. Using Lemma 1.2.1, we get $\|\Phi_\rho(t,0)\|^2 \leq \gamma_1^2 =: \delta$ and

$$\begin{aligned}
\sigma_T &= \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \mathcal{E}\left\{\frac{1}{T}\int_0^T x^T(t)C^T(\rho(t))C(\rho(t))x(t)dt\right\} \\
&\leq \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \frac{1}{T}\,\mathcal{E}\left\{\int_0^T \|C(\rho(t))\|_F^2\,\|\Phi_\rho(t,0)\|^2\,\|\bar{x}_0\|^2\,dt + \int_0^T tr\left[C(\rho(t))Q(t)C^T(\rho(t))\right]dt\right\} \\
&\leq \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \left\{\frac{\delta}{T}\,tr\left[C(\rho(t))tr\left(\bar{x}_0\bar{x}_0^T\right)C^T(\rho(t))\right] + tr\left[C(\rho(t))Q(t)C^T(\rho(t))\right]\right\} \\
&\leq \max_{\rho \in \mathcal{P}} tr\left\{C(\rho)\left[\frac{\delta}{T}\,tr\left(\bar{x}_0\bar{x}_0^T\right)I + Y^{-1}\right]C^T(\rho)\right\}.
\end{aligned}$$

Note that δ is independent of T. Taking limit $T \to \infty$ on both sides of above equation leads to

$$\sigma_\infty \leq \max_{\rho \in \mathcal{P}} tr\left[Y^{-1}C^T(\rho)C(\rho)\right].$$

The above inequality is true for any $Y \in \mathcal{Y}$ with $Y < Q_0^{-1}$, so that

$$\sigma_\infty \leq \inf_{Y < Q_0^{-1},\, Y \in \mathcal{Y}} \max_{\rho \in \mathcal{P}} tr\left[Y^{-1}C^T(\rho)C(\rho)\right].$$

∎

Remark 1.4.1 *If the system is linear time-invariant, then* α *in Theorem 1.4.1 and* β *in Theorem 1.4.2 are the best upper bounds. In fact*

$$\|C(sI - A)^{-1}B\|_2^2 = \alpha = \beta,$$

so both bounds are the actual cost for LTI systems.

For LPV systems, Theorem 1.4.1 and Theorem 1.4.2 give two LQG performance bounds, which may not be the same. For a particular problem, either bound can be better than the other.

Example 1.4.1 *Given* $\mathcal{P} = [-1, 1]$, *consider a LPV system*

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. $\bar{x}_0 = 0$ *and* $Q_0 = 0$, $d(t)$ *is a white noise with intensity* 1. *Let* $A(\rho) = A_0 + \rho A_1$, $B(\rho)$ *and* $C(\rho)$ *are constant matrices.*

Solution: Case 1: Suppose

$$A_0 = \begin{bmatrix} -1.0 & 0.8 \\ 0 & -2.0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad C = \begin{bmatrix} 1.342 & 0.447 \\ 0.447 & 0.894 \end{bmatrix}.$$

Then from Theorem 1.4.1 and Theorem 1.4.2, we get $\alpha = 1.64 < \beta = 1.98$.

Case 2: By exchanging B and C matrices, that is

$$C = \begin{bmatrix} 1.342 & 0.447 \\ 0.447 & 0.894 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix},$$

we get $\beta = 1.67 < \alpha = 2.04$. ∎

In Chapter 2, we will use analysis results given by Theorem 1.4.1 and Theorem 1.4.2 to solve state-feedback control and state estimation problems for LPV systems.

1.5 Convexity and Complexity of Analysis Results

In this section, we discuss general properties of two analysis results given by Theorem 1.4.1 and Theorem 1.4.2. Our goal is to reformulate the existing conditions in the form of linear matrix inequalities (LMIs) [PacZPB], [BoyEFB], which have advantages from a computational point of view.

First, define the matrix valued functions

$$\begin{aligned} Ric_X^2(X,\rho) &:= \begin{bmatrix} A(\rho)X + XA^T(\rho) & XC^T(\rho) \\ C(\rho)X & -I_{n_e} \end{bmatrix}, \\ Ric_Y^2(Y,\rho) &:= \begin{bmatrix} YA(\rho) + A^T(\rho)Y & YB(\rho)V^{\frac{1}{2}} \\ V^{\frac{1}{2}}B^T(\rho)Y & -I_{n_d} \end{bmatrix}. \end{aligned}$$

Note that $Ric_X^2(X,\rho) < 0$ is nothing but the stability condition (1.2.2) by Schur complement. Similarly $Ric_Y^2(Y,\rho) < 0$ is equivalent to condition (1.2.3). But it is easy to see that while $Ric_X^2(X,\rho)$ is an affine function of its variable X, condition (1.2.2) is not. Similarly $Ric_Y^2(Y,\rho)$ is affine on Y.

Lemma 1.5.1 *For fixed $\bar{\rho} \in \mathcal{P}$, the scalar valued function $\lambda_{max}\left[Ric_X^2(X,\bar{\rho})\right] : \mathcal{S}_+^{n\times n} \to \mathbf{R}$ is convex of its variable X, and $\lambda_{max}\left[Ric_Y^2(Y,\bar{\rho})\right] : \mathcal{S}_+^{n\times n} \to \mathbf{R}$ is convex function of Y.*

Proof: $\lambda_{max}(\cdot)$ is a convex function of its argument, and $Ric_X^2(X,\bar{\rho})$, $Ric_Y^2(Y,\bar{\rho})$ are affine functions of X, Y respectively, so the convexity of both functions is clear. ∎

Remark 1.5.1 *By Schur complement, the set $\mathcal{X}$ defined in Theorem 1.4.1 can be expressed with Ric_X^2,*

$$\mathcal{X} = \bigcap_{\rho\in\mathcal{P}} \left\{ X \in \mathcal{S}_+^{n\times n} : \ Ric_X^2(X,\rho) < 0 \right\}.$$

Hence $\mathcal{X}$ is a convex set. Similarly, $\mathcal{Y}$ is convex set of Y.

Next we will state a generalization of the result from [KhaR] which is about the convexity of function $f(X,Y) = tr\left(R^T X^T Y^{-1} X R\right)$.

Lemma 1.5.2 *Let $R \in \mathbf{R}^{m\times m}$ be invertible, define a function $f(X,Y) = tr\left(R^T X^T Y^{-1} X R\right)$ where $X \in \mathbf{R}^{n\times m}$ and $Y \in \mathcal{S}_+^{n\times n}$, Then f is convex function of X, Y jointly.*

Proof: For the proof of special case $R = I$, see [KhaR]. Generally, define $\tilde{X} = XR$, then

$$f(X,Y) = tr\left(R^T X^T Y^{-1} X R\right) = tr\left(\tilde{X}^T Y^{-1} \tilde{X}\right) =: \tilde{f}(\tilde{X}, Y)$$

Function $\tilde{f}$ is convex of $\tilde{X}, Y$. For any $\tilde{X}_1, \tilde{X}_2 \in \mathbf{R}^{n\times m}$ and $Y_1, Y_2 \in \mathcal{S}_+^{n\times n}$, we have from the definition of convex function that

$$\begin{aligned} & tr\left[\left(\lambda \tilde{X}_1 + (1-\lambda)\tilde{X}_2\right)^T (\lambda Y_1 + (1-\lambda) Y_2)^{-1} \left(\lambda \tilde{X}_1 + (1-\lambda)\tilde{X}_2\right)\right] \\ & \leq \; \lambda \, tr\left(\tilde{X}_1^T Y_1^{-1} \tilde{X}_1\right) + (1-\lambda)\, tr\left(\tilde{X}_2^T Y_2^{-1} \tilde{X}_2\right). \end{aligned}$$

Substitute $\tilde{X}_1, \tilde{X}_2$ back by X_1, X_2, we get the inequality which shows clearly the convexity of function f. ∎

With above two lemmas in mind, now we can study the computational issues of LQG performance bounds α and β.

1.5.1 Simplification of Analysis Results

Under some simplifying assumptions on state-space data, the computation of α, β become finite-dimensional convex programs. By finite-dimensional, we mean finite number of objectives and constraints. The assumptions we need are:

- $\mathcal{P}$ is a convex polytope in $\mathbf{R}^s$ with its (finite) set of extreme points denoted by $\mathcal{V}$,
- A, B and C are all affine functions of ρ, that is

$$A(\rho) = A_0 + \sum_{i=1}^{s} \rho_i A_i, \tag{1.5.1.a}$$

$$B(\rho) = B_0 + \sum_{i=1}^{s} \rho_i B_i, \tag{1.5.1.b}$$

$$C(\rho) = C_0 + \sum_{i=1}^{s} \rho_i C_i. \tag{1.5.1.c}$$

Then we have the following result:

Theorem 1.5.1 *Given $\mathcal{P}$, $\mathcal{V}$, A, B, C as above. Define*

$$\mathcal{X}^{\mathcal{V}} := \left\{ X \in \mathcal{S}_+^{n\times n} : \max_{v\in\mathcal{V}} \lambda_{max}\left[A(v)X + XA^T(v) + XC^T(v)C(v)X\right] < 0 \right\},$$

and $\alpha^{\mathcal{V}}$ as

$$\alpha^{\mathcal{V}} := \inf_{X\in\mathcal{X}^{\mathcal{V}}} \max_{v\in\mathcal{V}} tr\left[X^{-1}B(v)VB^T(v)\right].$$

Then $\mathcal{X} = \mathcal{X}^{\mathcal{V}}$ and $\alpha = \alpha^{\mathcal{V}}$, where $\mathcal{X}$, α are defined in Theorem 1.4.1.

Remark 1.5.2 *$\mathcal{X}^{\mathcal{V}}$ is a finite constraint set of X as $\mathcal{V}$ is finite element set. Recall a well-known lemma from convex analysis: for convex function $f(\rho)$ over convex polytope $\mathcal{P}$,*

$$\max_{\rho\in\mathcal{P}} f(\rho) = \max_{v\in\mathcal{V}} f(v).$$

Proof of Theorem 1.5.1: Obviously, $\mathcal{X} \subset \mathcal{X}^{\mathcal{V}}$. Using Schur complement argument, we rewrite sets $\mathcal{X}$ and $\mathcal{X}^{\mathcal{V}}$ as

$$\begin{aligned} \mathcal{X} &= \left\{X \in \mathcal{S}_+^{n\times n} : \ Ric_X^2(X,\rho) < 0, \ \forall\, \rho \in \mathcal{P}\right\}, \\ \mathcal{X}^{\mathcal{V}} &= \left\{X \in \mathcal{S}_+^{n\times n} : \ Ric_X^2(X,v) < 0, \ \forall\, v \in \mathcal{V}\right\}. \end{aligned}$$

Let $X \in \mathcal{X}^{\mathcal{V}}$. By the assumptions of affine dependence of A and C on parameter ρ, we conclude that $\lambda_{max}\left[Ric_X^2(X,\rho)\right]$ is a convex function of ρ for fixed X. From Remark 1.5.2, it is clear that for all $\rho \in \mathcal{P}$

$$\lambda_{max}\left[Ric_X^2(X,\rho)\right] \le \max_{\rho\in\mathcal{P}} \lambda_{max}\left[Ric_X^2(X,\rho)\right] = \max_{v\in\mathcal{V}} \lambda_{max}\left[Ric_X^2(X,v)\right] < 0,$$

which implies that $X \in \mathcal{X}$. So $\mathcal{X} = \mathcal{X}^{\mathcal{V}}$.

For fixed $X \in \mathcal{X}$, define function $f(\rho) := tr\left[V^{\frac{1}{2}}B^T(\rho)X^{-1}B(\rho)V^{\frac{1}{2}}\right]$. Note that f is a convex function of B by Lemma 1.5.2, and B is in the affine form of parameter ρ, so f is a convex function of ρ. It is a known fact (see Remark 1.5.2) that a convex function achieves its maximum at extreme points. Given convex polytope $\mathcal{P}$, its finite set of extreme points $\mathcal{V}$ and convex function $f(\rho)$, we have

$$\max_{\rho\in\mathcal{P}} f(\rho) = \max_{v\in\mathcal{V}} f(v).$$

Combined with previous result $\mathcal{X} = \mathcal{X}^{\mathcal{V}}$, we finally get $\alpha = \alpha^{\mathcal{V}}$. ∎

Theorem 1.5.1 shows that with affine parameter dependence assumption on the state-space data, the bound α becomes a finite-dimensional convex problem. A similar result for the bound β is given below:

Theorem 1.5.2 *Given a convex polytope $\mathcal{P}$ with $\mathcal{V}$ as its finite set of extreme points, and affine assumptions in (1.5.1.) on the state-space data. Define*

$$\mathcal{Y}^{\mathcal{V}} := \left\{Y \in \mathcal{S}_+^{n\times n} : \ \max_{v\in\mathcal{V}} \lambda_{max}\left[YA(v) + A^T(v)Y + YB(v)VB^T(v)Y\right] < 0\right\},$$

and $\beta^{\mathcal{V}}$ as

$$\beta^{\mathcal{V}} := \inf_{Y<Q_0^{-1},\ Y\in\mathcal{Y}^{\mathcal{V}}} \ \max_{v\in\mathcal{V}} tr\left[Y^{-1}C^T(v)C(v)\right].$$

Then $\mathcal{Y} = \mathcal{Y}^{\mathcal{V}}$ and $\beta = \beta^{\mathcal{V}}$, where $\mathcal{Y}$, β are defined in Theorem 1.4.2.

Proof: The proof is similar to the one for Theorem 1.5.1. ∎

1.5.2 Complexity Study

In general, the bounds α, β are hard to compute because both involve an expression of the form $\max_{\rho\in\mathcal{P}}$ to be minimized over the set $\mathcal{X}$ (or $\mathcal{Y}$), which includes infinite number of constraints. Instead of computing α and β directly, we may minimize bounds for α, β as follows.

Theorem 1.5.3 *Suppose $W_1 \geq B(\rho)VB^T(\rho)$, $W_2 \geq C^T(\rho)C(\rho)$ for all $\rho \in \mathcal{P}$, then*

$$\begin{aligned} \alpha &\leq \inf_{X\in\mathcal{X}} tr\left(X^{-1}W_1\right) =: \alpha_{sub}, \\ \beta &\leq \inf_{Y<Q_0^{-1},\ Y\in\mathcal{Y}} tr\left(Y^{-1}W_2\right) =: \beta_{sub}, \end{aligned}$$

where α, β are defined in Theorem 1.4.1 and Theorem 1.4.2 respectively.

Proof: Obviously,

$$\begin{aligned} \alpha &= \inf_{X\in\mathcal{X}} \max_{\rho\in\mathcal{P}} tr\left[X^{-1}B(\rho)VB^T(\rho)\right] = \inf_{X\in\mathcal{X}} \max_{\rho\in\mathcal{P}} tr\left[X^{-\frac{1}{2}}B(\rho)VB^T(\rho)X^{-\frac{1}{2}}\right] \\ &\leq \inf_{X\in\mathcal{X}} tr\left(X^{-\frac{1}{2}}W_1X^{-\frac{1}{2}}\right) = \inf_{X\in\mathcal{X}} tr\left(X^{-1}W_1\right) = \alpha_{sub}. \end{aligned}$$

Similarly, we can prove $\beta \leq \beta_{sub}$. ∎

By Lemma 1.5.2, the computations of the bounds α_{sub}, β_{sub} are convex optimization problems of one objective over infinite number of LMI constraints ($X \in \mathcal{X}$, or $Y \in \mathcal{Y}$). These bounds can be computed approximately with finite constraints by gridding the compact set $\mathcal{P}$. Furthermore, α_{sub} and β_{sub} are not affine functions of their variables X, Y respectively, but this can be cured by appending another variable to the optimizations.

Corollary 1.5.1 *Define two functions*

$$\psi(X,Z) := \begin{bmatrix} Z & I \\ I & X \end{bmatrix}, \quad \phi(Y,Z) = \begin{bmatrix} Z & I \\ I & Y \end{bmatrix}.$$

Then the α_{sub} and β_{sub} given in Theorem 1.5.3 can be rewritten as

$$\begin{aligned} \alpha_{sub} &= \inf_{\substack{X\in\mathcal{X} \\ \psi(X,Z)\geq 0}} tr(ZW_1), \\ \beta_{sub} &= \inf_{\substack{Y<Q_0^{-1},\ Y\in\mathcal{Y} \\ \phi(Y,Z)\geq 0}} tr(ZW_2). \end{aligned}$$

Proof: Let $\tilde{\alpha} := \inf_{\substack{X\in\mathcal{X} \\ \psi(X,Z)\geq 0}} tr(ZW_1)$. The constraint $\psi(X,Z) \geq 0$ simply implies $Z \geq X^{-1}$. So we have

$$\alpha_{sub} = \inf_{X\in\mathcal{X}} tr(X^{-1}W_1) \leq \inf_{\substack{X\in\mathcal{X} \\ Z\geq X^{-1}}} tr(ZW_1) = \tilde{\alpha}.$$

On the other hand, for any $\epsilon > 0$, there exists a matrix $\hat{X} \in \mathcal{X}$, such that

$$tr(\hat{X}^{-1}W_1) \leq \alpha_{sub} + \epsilon.$$

Choosing $\hat{Z} = \hat{X}^{-1}$ which satisfies the condition $Z \geq X^{-1}$, then

$$\tilde{\alpha} = \inf_{\substack{X\in\mathcal{X} \\ \psi(X,Z)\geq 0}} tr(ZW_1) \leq tr(\hat{Z}W_1) \leq \alpha_{sub} + \epsilon.$$

Note above inequality is true for arbitrarily small ϵ, it must have $\tilde{\alpha} \leq \alpha_{sub}$. So we get

$$\alpha_{sub} = \tilde{\alpha} = \inf_{\substack{X\in\mathcal{X} \\ \psi(X,Z)\geq 0}} tr(ZW_1).$$

The proof for β_{sub} can be done similarly. ∎

Remark 1.5.3 *After this, we will only mention such a standard transformation to LMI formulation without proof.*

We have shown the procedure to compute the LQG performance bounds for LPV systems. It is an LMI optimization problem, though it generally requires gridding of the parameter space. The resulting performance bounds can be computed efficiently by LMI optimization methods, such as projective method [NemG], [GahNLC], method of centers [BoyE], [NekF].

1.6 Comparison with Previous Results

We should point out that Theorem 1.4.1 and Theorem 1.4.2 are motivated by [PetH] and [BerH4]. In these references, they develop clever matrix bounds for structured real parameter uncertainty, and then solve the resulting over-bounded Lyapunov or Riccati equality using conventional techniques. Here, we use an inequality formulation which leads to a less conservative bound for the LQG performance.

In this section, we concentrate on a particular bound, the Petersen-Hollot bound, as generalized in [BerH4]. In the following theorem, the bound from [BerH4] is denoted $\alpha_{\mathcal{B}}$.

Theorem 1.6.1 *Given a compact set* $\mathcal{P} := [-\delta_1, \delta_1]\times[-\delta_2, \delta_2]\times\cdots\times[-\delta_s, \delta_s]$, *where* $\{\delta_i\}_{i=1}^{s}$ *are non-negative, and the LPV system in (1.4.1) and (1.4.2.). Suppose* $A(\rho) := A_0 + \sum_{i=1}^{s} \rho_i A_i$ *is affine function of* ρ, *and* B, C *are constant with* $C^T C > 0$. *For any positive numbers* $\{\gamma_i\}_{i=1}^{s}$, *define a set* $\mathcal{X}_{\mathcal{B}}^{\gamma}$ *as*

$$\mathcal{X}_{\mathcal{B}}^{\gamma} := \Big\{X \in \mathcal{S}_{+}^{n\times n} : \; A(\rho)X + XA^T(\rho) + XC^T CX + \hat{\Delta}(\rho) = 0,$$
$$\hat{\Delta}(\rho) := \sum_{i=1}^{s} \delta_i \left(\gamma_i XX + \frac{1}{\gamma_i} A_i A_i^T\right) - \sum_{i=1}^{s} \rho_i \left(A_i X + XA_i^T\right), \; \forall \rho \in \mathcal{P}\Big\}$$

where $\gamma := [\gamma_1 \quad \cdots \quad \gamma_s]^T$. *Furthermore, let*

$$\alpha_{\mathcal{B}} := \inf_{\gamma} \inf_{X \in \mathcal{X}_{\mathcal{B}}^{\gamma}} tr\left(X^{-1} BVB^T\right).$$

Then $\alpha \leq \alpha_{\mathcal{B}}$, *where* α *is defined in Theorem 1.4.1.*

Remark 1.6.1 *Although the constraints in the set* $\mathcal{X}_{\mathcal{B}}^{\gamma}$ *seem to depend on the parameters, they are actually not. The element of the set* $\mathcal{X}_{\mathcal{B}}^{\gamma}$ *is the positive definite matrix which satisfies equation*

$$A_0 X + XA_0^T + X\left(C^T C + \sum_{i=1}^{s} \delta_i \gamma_i I\right) X + \sum_{i=1}^{s} \frac{\delta_i}{\gamma_i} A_i A_i^T = 0.$$

Proof of Theorem 1.6.1: First note, for any $\rho \in \mathcal{P}$, positive numbers $\{\gamma_i\}_{i=1}^{s}$ and $X \in \mathcal{S}_{+}^{n\times n}$, we have

$$\hat{\Delta}(\rho) = \sum_{i=1}^{s} \delta_i \left(\gamma_i XX + \frac{1}{\gamma_i} A_i A_i^T\right) - \sum_{i=1}^{s} \rho_i \left(A_i X + XA_i^T\right) \geq 0.$$

The set $\mathcal{X}$ can be rewritten as

$$\mathcal{X} = \left\{X \in \mathcal{S}_{+}^{n\times n} : \; A(\rho)X + XA^T(\rho) + XC^T CX + \Delta(\rho) = 0, \; \Delta(\rho) > 0, \; \forall \rho \in \mathcal{P}\right\},$$

also define an intermediate set $\tilde{\mathcal{X}}$ to be

$$\tilde{\mathcal{X}} = \left\{X \in \mathcal{S}_{+}^{n\times n} : \; A(\rho)X + XA^T(\rho) + XC^T CX + \Delta(\rho) = 0, \; \Delta(\rho) \geq 0, \; \forall \rho \in \mathcal{P}\right\}.$$

Note that $\mathcal{X}_{\mathcal{B}}^{\gamma} \subset \tilde{\mathcal{X}}$ for all positive vector γ. Let $\tilde{\alpha} := \inf_{X \in \tilde{\mathcal{X}}} tr\left(X^{-1} BVB^T\right)$, then α, $\tilde{\alpha}$ and $\alpha_{\mathcal{B}}$ are minimizations of the same function but over different sets ($\mathcal{X}$, $\tilde{\mathcal{X}}$ and $\mathcal{X}_{\mathcal{B}}$ respectively). For any $\tilde{X} \in \tilde{\mathcal{X}}$, define a sequence of matrix $X_k := \frac{k}{k+1}\tilde{X}, \; k = 1, 2, \cdots$, then

$$A(\rho)\tilde{X} + \tilde{X}A^T(\rho) + \tilde{X}C^T C\tilde{X} + \Delta(\rho) = 0 \tag{1.6.1}$$

with $\Delta(\rho) \geq 0$ for all $\rho \in \mathcal{P}$. Substitute $\tilde{X}$ by $\frac{k+1}{k}X_k$ in equation (1.6.1), we get

$$A(\rho)X_k + X_k A^T(\rho) + X_k C^T C X_k = -\frac{1}{k} X_k C^T C X_k - \frac{k}{k+1}\Delta(\rho) < 0$$

for any k. So $X_k \in \mathcal{X},\ k = 1, 2, \cdots$ and $\lim_{k\to\infty} X_k = \tilde{X}$. This proves that $closure(\mathcal{X}) = \tilde{\mathcal{X}}$ and

$$\alpha = \inf_{X\in\mathcal{X}} tr\left(X^{-1}BVB^T\right) = \inf_{X\in\tilde{\mathcal{X}}} tr\left(X^{-1}BVB^T\right).$$

Defining $\alpha_{\mathcal{B}}^{\gamma} := \inf_{X\in\mathcal{X}_{\mathcal{B}}^{\gamma}} tr\left(X^{-1}BVB^T\right)$ for given γ, then

$$\alpha \leq \alpha_{\mathcal{B}}^{\gamma}.$$

Taking the infimum over all possible γ and leaving $\alpha \leq \alpha_{\mathcal{B}}$ as desired. ∎

Remark 1.6.2 *The same conclusion can be established for other kinds of bound in [BerH4].*

We can easily derive the dual result for the LQG performance bound β. The following example (adopted from [BerH4]) shows that in some cases, $\alpha < \alpha_{\mathcal{B}}$. So our analysis results generally lead to less conservative bounds compared with over-bounding equation approach. For a specific problem, it could be that they provide better bounds than previous ones.

Example 1.6.1 *A pair of nominally uncoupled oscillators with uncertain coupling is described by*

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_0 + \rho(t)A_1 & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$$

where

$$A_0 = \begin{bmatrix} -\nu & \omega_1 & 0 & 0 \\ -\omega_1 & -\nu & 0 & 0 \\ 0 & 0 & -\nu & \omega_2 \\ 0 & 0 & -\omega_2 & -\nu \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix},$$

with $\nu = 0.2$, $\omega_1 = 0.2$ *and* $\omega_2 = 1.8$. $B = C = diag\,(2.236, 1.495, 1.0, 0.669)$. *Furthermore, we assume* $d(t)$ *is white noise with intensity* $V = I_4$.

Solution: Given $\delta_1 > 0$. For $|\rho| \le \delta_1$ and any positive γ_1, we have the Riccati equation

$$A_0X + XA_0^T + X\left(C^TC + \delta_1\gamma_1 I_4\right)X + \frac{\delta_1}{\gamma_1}A_1A_1^T = 0. \qquad (1.6.2)$$

If equation (1.6.2) is solvable with $X > 0$, then one performance bound is given by $\alpha_{\mathcal{B}} = tr\left(X^{-1}BVB^T\right)$. But equation (1.6.2) may not have positive definite solution for a given γ_1. For computational purpose, the minimization of $\alpha_{\mathcal{B}}$ is conducted by a simple one-dimensional search for best γ_1 first, then solve for resulted Riccati equation. On the other hand, α is computed by convex optimization. The performance bounds over different parameter intervals from both approaches are plotted in Figure 1.1. ∎

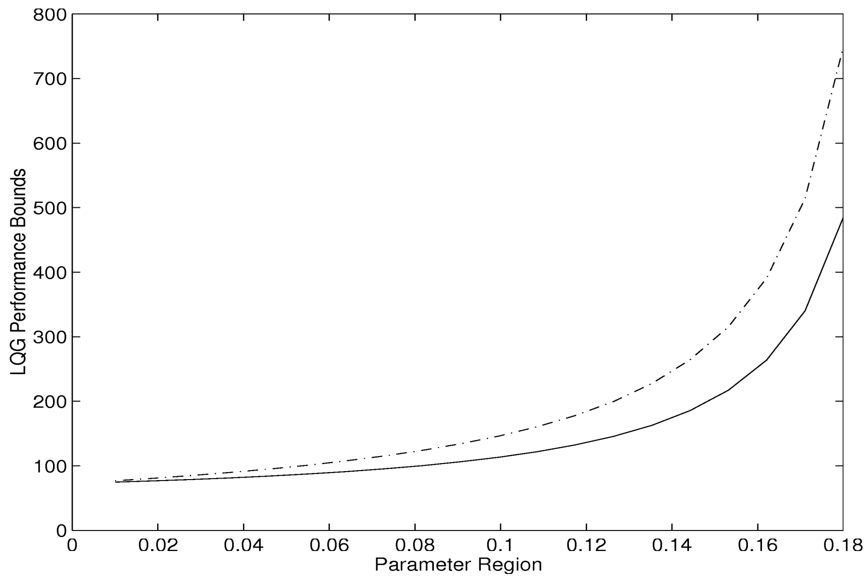

Figure 1.1: LQG performance bounds from Petersen-Hollot over-bounded equation (dash-dot line) and inequality approach (solid line).

Last, we would like to compare the computational costs of two methods. The computation of minimal bound $\alpha_{\mathcal{B}}$ involves search for the best γ. Currently, we do not know how to search over a vector space of γ globally. So the bound $\alpha_{\mathcal{B}}$ can only be calculated approximately by the gridding method. But bound α can be computed exactly through convex optimization over 2^s LMI constraints. The computational scheme for α is systematic and very attractive because of recently developed techniques to handle LMI optimization problem. Furthermore, This approach always provides less conservative bound for LQG performance compared with over-bounding method we mentioned above.

Chapter 2

Control of LPV Systems with LQG Performance

In this chapter, we solve state-feedback, state-estimation and output-feedback control problems using the analysis results for LQG performance described in Theorem 1.4.1 and Theorem 1.4.2.

In §2.1, we define the feedback problem. In §2.2, we study parameter-dependent and robust state-feedback control of LPV systems. In §2.3 we discuss the state estimation problem for LPV systems, which is the dual of state-feedback problem. In §2.4, we design two output-feedback controllers based on the separation principle and the results established in §2.2 and §2.3. Explicit LQG performance bounds are derived for both cases. In §2.5, we give a computational scheme to compute the LQG performance bounds.

2.1 Quadratic LQG Performance Problem

Before formulating the problem we are interested in, we will define open-loop LPV systems for control synthesis and the class of parameter-dependent controllers.

Definition 2.1.1 Open-Loop LPV System for LQG control
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, consider the open-loop LPV system $\Sigma_{\mathcal{P}}$

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & D_{11}(\rho(t)) & D_{12}(\rho(t)) \\ C_2(\rho(t)) & D_{21}(\rho(t)) & D_{22}(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}, \tag{2.1.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$*,* $x(t), \dot{x}(t) \in \mathbf{R}^n$*,* $d(t) \in \mathbf{R}^{n_d}$*,* $e(t) \in \mathbf{R}^{n_e}$*,* $u(t) \in \mathbf{R}^{n_u}$ *and* $y(t) \in \mathbf{R}^{n_y}$*. All of the state-space matrices are continuous functions of* ρ *with appropriate dimensions. The statistics of stochastic variables* $x(0)$ *and* $d(t)$ *are given by (1.4.2.).*

For the purpose of simplification, the following assumptions are made on the state-space data of LPV systems:

(A1) $D_{11}(\rho) = 0_{n_e \times n_d}$,

(A2) $D_{22}(\rho) = 0_{n_y \times n_u}$,

(A3) $D_{12}(\rho)$ is of full column rank for all $\rho \in \mathcal{P}$,

(A4) $D_{21}(\rho)$ is of full row rank for all $\rho \in \mathcal{P}$.

Assumption $(A1)$ is sufficient to render the feed through from $d \to e$ strictly proper. Assumption $(A2)$ can be relaxed easily. Under assumptions $(A3)$ and $(A4)$, the D_{12} and D_{21} terms of above LPV system can be rewritten as $[0 \;\; I]^T$ and $[0 \;\; I]$ through norm-preserving transformations on disturbance/error and invertible transformations on input/output signal.

Definition 2.1.2 Simplified Open-Loop LPV System for LQG Control
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$*, and the open-loop LPV system* $\Sigma_{\mathcal{P}}$ *in Definition 2.1.1 with Assumptions* $(A1) - (A4)$ *hold, then the simplified open-loop LPV system can be written as*

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ C_{11}(\rho(t)) & 0 & 0 & 0 \\ C_{12}(\rho(t)) & 0 & 0 & I_{n_{d2}} \\ C_2(\rho(t)) & 0 & I_{n_{e2}} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix}, \qquad (2.1.2)$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$*. The statistics of stochastic variables* $x(0)$ *and* $d(t)$ *satisfy*

$$\mathcal{E}\{x(0)\} := \bar{x}_0, \qquad (2.1.3.a)$$

$$\mathcal{E}\left\{[x(0) - \bar{x}_0]\,[x(0) - \bar{x}_0]^T\right\} := Q_0, \qquad (2.1.3.b)$$

$$\mathcal{E}\left\{\begin{bmatrix} d_1(t_1) \\ d_2(t_1) \end{bmatrix} \begin{bmatrix} d_1^T(t_2) & d_2^T(t_2) \end{bmatrix}\right\} := \begin{bmatrix} V_{11}(\rho(t)) & V_{12}(\rho(t)) \\ V_{12}^T(\rho(t)) & V_{22}(\rho(t)) \end{bmatrix} \delta(t_1 - t_2), \qquad (2.1.3.c)$$

where $Q_0 > 0$*,* $V_{11}(\rho) \geq 0$*,* $V_{22}(\rho) > 0$ *and* $V_{11}(\rho) - V_{12}(\rho)V_{22}^{-1}(\rho)V_{12}^T(\rho) \geq 0$ *for all* $\rho \in \mathcal{P}$*.*

The class of finite-dimensional, linear parameter-dependent controllers is defined as:

Definition 2.1.3 Parameter-Dependent Controller
Let $K_{\mathcal{P}}$ denote a m-dimensional, parameter-dependent linear feedback controller, with the continuous functions $(A_K, B_K, C_K) : \mathbf{R}^s \to (\mathbf{R}^{m\times m}, \mathbf{R}^{m\times m_y}, \mathbf{R}^{m_u\times m})$. The dynamics of controller $K_{\mathcal{P}}$ are

$$\begin{bmatrix} \dot{x}_K(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_K(\rho(t)) & B_K(\rho(t)) \\ C_K(\rho(t)) & 0_{n_u\times n_y} \end{bmatrix} \begin{bmatrix} x_K(t) \\ y(t) \end{bmatrix}, \tag{2.1.4}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$, x_K is the m-dimensional controller states.

Define $x^T_{\text{clp}}(t) := \begin{bmatrix} x^T(t) & x^T_K(t) \end{bmatrix}$, $e^T(t) := \begin{bmatrix} e^T_1(t) & e^T_2(t) \end{bmatrix}$ and $d^T(t) := \begin{bmatrix} d^T_1(t) & d^T_2(t) \end{bmatrix}$. The closed-loop state-space data of LPV system using parameter-dependent controller is explicitly given by

$$\begin{bmatrix} \dot{x}_{\text{clp}}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{\text{clp}}(\rho(t)) & B_{\text{clp}}(\rho(t)) \\ C_{\text{clp}}(\rho(t)) & D_{\text{clp}}(\rho(t)) \end{bmatrix} \begin{bmatrix} x_{\text{clp}}(t) \\ d(t) \end{bmatrix}, \tag{2.1.5}$$

where

$$A_{\text{clp}}(\rho) := \begin{bmatrix} A(\rho) & B_2(\rho)C_K(\rho) \\ B_K(\rho)C_2(\rho) & A_K(\rho) \end{bmatrix}, \tag{2.1.6.a}$$

$$B_{\text{clp}}(\rho) := \begin{bmatrix} B_{11}(\rho) & B_{12}(\rho) \\ 0 & B_K(\rho) \end{bmatrix}, \tag{2.1.6.b}$$

$$C_{\text{clp}}(\rho) := \begin{bmatrix} C_{11}(\rho) & 0 \\ C_{12}(\rho) & C_K(\rho) \end{bmatrix}, \tag{2.1.6.c}$$

$$D_{\text{clp}}(\rho) := 0. \tag{2.1.6.d}$$

The closed-loop system has $D_{\text{clp}} = 0$ because the "D" terms in open-loop LPV system (D_{11}) and controller (D_K) are forced to be zero, which is sufficient to render finite LQG performance for closed-loop system. Now the LQG control synthesis problem is defined as follows:

Definition 2.1.4 Quadratic LQG Performance Problem

Given the open-loop LPV system $\Sigma_{\mathcal{P}}$ in (2.1.2) and (2.1.3.), and parameter-dependent controller $K_{\mathcal{P}}$ in (2.1.4), the LQG performance is

$$\sigma_T = \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \mathcal{E} \left\{ \frac{1}{T} \int_0^T \left[e_1^T(t) e_1(t) + e_2^T(t) e_2(t) \right] dt \right\},$$

and $\sigma_\infty = \lim_{T \to \infty} \sigma_T$. Then the Quadratic LQG Performance Problem is: determine the quadratically stabilizing controller $K_{\mathcal{P}}$ such that the LQG performance is bounded above and the bound is minimized.

The scalar σ_∞ represents the "worst" case cost for LQG performance over infinite horizon. It depends on controller, but not on a particular trajectory $\rho \in \mathcal{F}_{\mathcal{P}}$. We would like to design linear parameter-dependent controllers that approximately "minimize" the bound of this criterion, and derive a computable upper bound to σ_∞. Also, for the special case of LTI plant, without parameter dependence, the approach should recover the standard $\mathcal{H}_2$ optimal control results.

Before solving Quadratic LQG Performance Problem, we will study state-feedback control problem with LQG performance and state-estimation problem in the next two sections.

2.2 State-Feedback Control Problem

For comparison, we state the LQG state feedback control problem for linear time-varying (LTV) systems first. In this section, we study LQG state-feedback control problem for LPV systems, which include both parameter-dependent and robust control cases. The problems deal with the existence of a stabilizing state-feedback controller, such that the closed-loop system has bounded LQG performance. Later on, the parameter-dependent state-feedback control result will be used to solve Quadratic LQG Performance Problem.

2.2.1 LQG Optimal Control for LTV Systems

The LQG optimal control problem has been extensively studied in many literatures as linear optimal regulator theory on both deterministic and stochastic setting [KwaS], [AndM]. Here we state the stochastic optimal regulator result. Note that in the case of LTV systems, the solution for optimal control is characterized by a matrix Riccati differential equation.

Consider the LTV system described by

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} A(t) & B_1(t) & B_2(t) \\ C_{11}(t) & 0 & 0 \\ C_{12}(t) & 0 & I_{n_{e2}} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}. \tag{2.2.1}$$

(Note: we have assumed $D_{12}(t) = [0 \;\; I_{n_{e2}}]^T$). The initial state $x(0)$ is a stochastic variable with $\mathcal{E}\{x(0)\} = \bar{x}_0$, $\mathcal{E}\left\{(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)^T\right\} = Q_0$. The intensity of white noise $d(t)$ is $V(t) \geq 0$. $x(0)$ is independent of $d(t)$.

The problem is to minimize the LQG performance criterion

$$\sigma_T^0 = \frac{1}{T}\mathcal{E}\left\{\int_0^T \left[e_1^T(t)e_1(t) + e_2^T(t)e_2(t)\right] dt + x^T(T)P_T x(T)\right\}$$

with a state feedback controller $u(t) = F^0(t)x(t)$. The result is given in the following theorem.

Theorem 2.2.1 *Given the LTV system in (2.2.1) and statistics of stochastic variables $x(0)$ and $d(t)$ as above. The optimal linear control law is*

$$u(t) = F^0(t)x(t),$$

where $F^0(t) = -\left[B_2^T(t)P(t) + C_{12}(t)\right]$. $P(t)$ is the non-negative definite solution of the differential Riccati equation

$$-\dot{P}(t) = P(t)\hat{A}(t) + \hat{A}^T(t)P(t) - P(t)B_2(t)B_2^T(t)P(t) + C_{11}^T(t)C_{11}(t)$$

with terminal condition $P(T) = P_T$ and $\hat{A}(t) := A(t) - B_2(t)C_{12}(t)$. The minimum of the criterion is

$$\sigma_T^0 = \frac{1}{T}\, tr\left[P(0)\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + \int_0^T P(t)B_1(t)V(t)B_1^T(t)\, dt\right].$$

Proof: The proof can be found in many references, for example, see [KwaS].

2.2.2 Quadratic State-Feedback Control for LPV Systems

We now consider quadratic state-feedback control for LPV systems. Recall that we have formulated in Theorem 1.3.1 the existence condition of quadratically stabilizing controller as an LMI of the open-loop state-space data of LPV systems (condition (1.3.2)), the LQG performance bound of the closed-loop system can be derived based on such a condition.

Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, consider LPV system

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_{11}(\rho(t)) & 0 & 0 \\ C_{12}(\rho(t)) & 0 & I_{n_{e2}} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}, \tag{2.2.2}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. $x(t) \in \mathbf{R}^n$, $d(t) \in \mathbf{R}^{n_d}$, $u(t) \in \mathbf{R}^{n_u}$, $e_1(t) \in \mathbf{R}^{n_{e1}}$ and $e_2(t) \in \mathbf{R}^{n_{e2}}$. The initial state $x(0)$ is stochastic variable and is independent of white noise $d(t)$ with

$$\mathcal{E}\left\{x(0)\right\} := \bar{x}_0, \tag{2.2.3.a}$$

$$\mathcal{E}\left\{(x(0)-\bar{x}_0)(x(0)-\bar{x}_0)^T\right\} := Q_0, \tag{2.2.3.b}$$

$$\mathcal{E}\left\{d(t_1)d^T(t_2)\right\} := V(\rho(t))\,\delta(t_1-t_2), \tag{2.2.3.c}$$

where $Q_0 > 0$ and $V(\rho) \geq 0$ for all $\rho \in \mathcal{P}$.

Theorem 2.2.2 *Given a compact set $\mathcal{P}$, and a quadratically stabilizable LPV system in (2.2.2) and (2.2.3.). Define a set $\mathcal{X}_{gs}$ as*

$$\mathcal{X}_{gs} := \left\{X \in \mathcal{S}_+^{n\times n} : \max_{\rho\in\mathcal{P}} \lambda_{max}\left[\hat{A}(\rho)X + X\hat{A}^T(\rho) - B_2(\rho)B_2^T(\rho) + XC_{11}^T(\rho)C_{11}(\rho)X\right] < 0\right\},$$

then

$$\sigma_\infty \leq \inf_{X\in\mathcal{X}_{gs}} \max_{\rho\in\mathcal{P}} tr\left[X^{-1}B_1(\rho)V(\rho)B_1^T(\rho)\right].$$

Proof: Suppose $F(\rho)$ is a state-feedback gain, then the closed-loop system is given by

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_F(\rho(t)) & B_1(\rho(t)) \\ C_F(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix},$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$, $A_F(\rho) := A(\rho) + B_2(\rho)F(\rho)$ and $C_F^T(\rho) := \left[C_{11}^T(\rho) \;\; C_{12}^T(\rho) + F^T(\rho)\right]$. Given $X \in \mathcal{S}_+^{n\times n}$, by Theorem 1.3.1, $X \in \mathcal{X}_{gs}$ if and only if there exists a function $F \in \mathcal{C}^0(\mathbf{R}^s, \mathbf{R}^{n_u\times n})$ such that

$$A_F(\rho)X + XA_F^T(\rho) + XC_F^T(\rho)C_F(\rho)X < 0$$

for all $\rho \in \mathcal{P}$. Using such a quadratically stabilizing control F, we get

$$\sigma_\infty \leq \inf_{X\in\mathcal{X}_{gs}} \max_{\rho\in\mathcal{P}} tr\left[X^{-1}B_1(\rho)V(\rho)B_1^T(\rho)\right]$$

from Theorem 1.4.1. ∎

Comparing with LTV optimal regulator result given in Theorem 2.2.1, we could see the similarity between them. But instead of solving differential Riccati equation for LTV case, we formulate an inequality about matrix X for LPV systems, from which the LQG performance bound is calculated. Note that X^{-1} in the bound plays similar role as P for linear optimal regulator.

2.2.3 Robust State-Feedback Control of LPV Systems

Here we deliberately ignore real-time information of parameter, and apply Theorem 1.4.1 and Theorem 1.4.2 to solve LQG performance oriented, robust state-feedback control synthesis problem.

Given a convex polytope $\mathcal{P} \subset \mathbf{R}^s$ with its finite set of extreme points denoted by $\mathcal{V}$, consider the open-loop LPV system

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & 0 & D_{12}(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}, \tag{2.2.4}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$, $x(t) \in \mathbf{R}^n$, $d(t) \in \mathbf{R}^{n_d}$, $u(t) \in \mathbf{R}^{n_u}$ and $e(t) \in \mathbf{R}^{n_e}$. The statistics of stochastic variables $x(0)$ and $d(t)$ are given by (1.4.2.). Assuming all of the state-space matrices are affine functions of ρ, that is

$$A(\rho) := A_0 + \sum_{i=1}^{s} \rho_i A_i, \tag{2.2.5.a}$$

$$B_1(\rho) := B_{10} + \sum_{i=1}^{s} \rho_i B_{1i}, \tag{2.2.5.b}$$

$$B_2(\rho) := B_{20} + \sum_{i=1}^{s} \rho_i B_{2i}, \tag{2.2.5.c}$$

$$C_1(\rho) := C_{10} + \sum_{i=1}^{s} \rho_i C_{1i}, \tag{2.2.5.d}$$

$$D_{12}(\rho) := D_{120} + \sum_{i=1}^{s} \rho_i D_{12i}. \tag{2.2.5.e}$$

Based on Theorem 1.3.2 about quadratic stabilizability using constant state-feedback, we get the synthesis result for robust state-feedback control as follows:

Theorem 2.2.3 *Given a convex polytope $\mathcal{P} \subset \mathbf{R}^s$ with its finite set of extreme points denoted by $\mathcal{V}$, and quadratically stabilizable LPV system by constant state-feedback in (2.2.4) and (1.4.2.) with affine function assumption in (2.2.5.). Define two sets $\mathcal{W}_{rob,1}^{\mathcal{V}}$, $\mathcal{W}_{rob,2}^{\mathcal{V}}$ as*

$$\begin{aligned}\mathcal{W}_{rob,1}^{\mathcal{V}} &:= \Big\{(X,R) \in \mathcal{S}_{+}^{n\times n} \times \mathbf{R}^{n_u\times n} : \max_{v\in\mathcal{V}} \lambda_{max}\Big[A(v)X + XA^T(v) \\ &\quad + B_2(v)R + R^T B_2^T(v) + (C_1(v)X + D_{12}(v)R)^T (C_1(v)X + D_{12}(v)R)\Big] < 0\Big\}, \\ \mathcal{W}_{rob,2}^{\mathcal{V}} &= \Big\{(Y,S) \in \mathcal{S}_{+}^{n\times n} \times \mathbf{R}^{n_u\times n} : \\ &\quad \max_{v\in\mathcal{V}} \lambda_{max}\Big[A(v)Y + YA^T(v) + B_2(v)S + S^T B_2^T(v) + B_1(v)VB_1^T(v)\Big] < 0\Big\}.\end{aligned}$$

(1) If there exist $(X,R) \in \mathcal{W}_{rob,1}^{\mathcal{V}}$, then

$$\sigma_\infty \le \alpha_{rob} := \inf_{(X,R)\in\mathcal{W}_{rob,1}^{\mathcal{V}}} \max_{v\in\mathcal{V}} tr\left[X^{-1}B_1(v)VB_1^T(v)\right]. \tag{2.2.6}$$

Picking feasible X, R which yields a cost close (as close as we want) to α_{rob}, then the robust stabilizing state-feedback control law is given by $u(t) = RX^{-1}x(t)$.

(2) If there exist $(Y,S) \in \mathcal{W}_{rob,2}^{\mathcal{V}}$ with $Y > Q_0$, then

$$\sigma_\infty \le \beta_{rob} := \inf_{Y>Q_0,\ (Y,S)\in\mathcal{W}_{rob,2}^{\mathcal{V}}} \max_{v\in\mathcal{V}} tr\left[(C_1(v)Y + D_{12}(v)S)\,Y^{-1}\,(C_1(v)Y + D_{12}(v)S)^T\right]. \tag{2.2.7}$$

Picking feasible Y, S which yields a cost close (as close as we want) to β_{sub}, then the robust stabilizing state-feedback controller is given by $u(t) = SY^{-1}x(t)$.

Proof: We will prove the second part of the theorem only, the first part can be proved similarly. Similar to $\mathcal{W}_{rob,2}^{\mathcal{V}}$, define an intermediate set $\mathcal{W}_{rob,2}$ as

$$\begin{aligned}\mathcal{W}_{rob,2} &= \Big\{(Y,S) \in \mathcal{S}_{+}^{n\times n} \times \mathbf{R}^{n_u\times n} : \\ &\quad \max_{\rho\in\mathcal{P}} \lambda_{max}\Big[A(\rho)Y + YA^T(\rho) + B_2(\rho)S + S^T B_2^T(\rho) + B_1(\rho)VB_1^T(\rho)\Big] < 0\Big\}.\end{aligned}$$

By affine parameter dependence assumption on state-space data, it is easy to show that $\mathcal{W}_{rob,2} = \mathcal{W}_{rob,2}^{\mathcal{V}}$. The assumption of quadratically stabilizable LPV system using constant state-feedback implies that the set $\mathcal{W}_{rob,2}^{\mathcal{V}}$ is non-empty (condition (1.3.7) in Theorem 1.3.2. If there exist $(Y,S) \in \mathcal{W}_{rob,2}$ with $Y > Q_0$, let $F := SY^{-1}$, then for all $\rho \in \mathcal{P}$

$$A_F(\rho)Y + YA_F^T(\rho) + B_1(\rho)VB_1^T(\rho) < 0 \tag{2.2.8}$$

with $A_F(\rho) = A(\rho) + B_2(\rho)F$ and $C_F(\rho) = C_1(\rho) + D_{12}(\rho)F$. Equation (2.2.8) is equivalent to

$$Y^{-1}A_F(\rho) + A_F^T(\rho)Y^{-1} + Y^{-1}B_1(\rho)VB_1^T(\rho)Y^{-1} < 0, \qquad \forall\, \rho \in \mathcal{P}.$$

From Theorem 1.4.2, we get the following using transformation $S = FY$

$$\begin{aligned} \sigma_\infty &\le \inf_{Y>Q_0,\ \max_{\rho\in\mathcal{P}}\lambda_{max}\left[A_F(\rho)Y+YA_F^T(\rho)+B_1(\rho)VB_1^T(\rho)\right]<0} \max_{\rho\in\mathcal{P}} tr\left[YC_F^T(\rho)C_F(\rho)\right] \\ &= \inf_{Y>Q_0,\ (Y,S)\in\mathcal{W}_{rob,2}} \max_{\rho\in\mathcal{P}} tr\left[\left(C_1(\rho)Y + D_{12}(\rho)S\right)Y^{-1}\left(C_1(\rho)Y + D_{12}(\rho)S\right)^T\right]. \end{aligned}$$

Define function $f(Y,S,\rho) := tr\left[\left(C_1(\rho)Y + D_{12}(\rho)S\right)Y^{-1}\left(C_1(\rho)Y + D_{12}(\rho)S\right)^T\right]$, then

$$f(Y,S,\rho) = tr\left[C_1(\rho)YC_1^T(\rho) + D_{12}(\rho)SC_1^T(\rho) + C_1(\rho)S^TD_{12}^T(\rho) + D_{12}(\rho)SY^{-1}S^TD_{12}^T(\rho)\right].$$

For fixed ρ, the first three terms of function $f(Y,S,\rho)$ are convex for sure. For the fourth term, define $\tilde{D}(\rho) = \left[D_{12}^T(\rho)D_{12}(\rho)\right]^{\frac{1}{2}}$ and note $\tilde{D}_{12}(\rho)$ is invertible, so that

$$tr\left[D_{12}(\rho)SY^{-1}S^TD_{12}^T(\rho)\right] = tr\left[\tilde{D}(\rho)SY^{-1}S^T\tilde{D}^T(\rho)\right],$$

which is convex function of Y, S by Lemma 1.5.2. So $f(Y,S,\rho)$ is indeed a convex function of Y, S for fixed ρ. Also by affine parameter dependence assumption, we can show that $f(Y,S,\rho)$ is convex function of parameter ρ for fixed Y, S. From Remark 2.5.2,

$$\max_{\rho\in\mathcal{P}} f(Y,S,\rho) = \max_{v\in\mathcal{V}} f(Y,S,v).$$

Finally, we get

$$\begin{aligned} \sigma_\infty &\le \inf_{Y>Q_0,\ (Y,S)\in\mathcal{W}_{rob,2}^{\mathcal{V}}} \max_{v\in\mathcal{V}} tr\left[\left(C_1(v)Y + D_{12}(v)S\right)Y^{-1}\left(C_1(v)Y + D_{12}(v)S\right)^T\right] \\ &= \beta_{rob} \end{aligned}$$

as desired. ∎

It is also easy to covert the convex optimization in equation (2.2.6) and (2.2.7) to LMI optimization problem by define two functions:

$$\psi(X,Z) := \begin{bmatrix} Z & I \\ I & X \end{bmatrix}, \qquad \phi(Y,S,Z,v) := \begin{bmatrix} Z & C_1(v)Y + D_{12}(v)S \\ YC_1^T(v) + S^TD_{12}^T(v) & Y \end{bmatrix}.$$

Then it is easy to show that

$$\alpha_{rob} = \inf_{\substack{(X,R)\in\mathcal{W}^{\mathcal{V}}_{rob,1} \\ \psi(X,Z)\geq 0}} \max_{v\in\mathcal{V}} tr\left[ZB_1(v)VB_1^T(v)\right],$$

$$\beta_{rob} = \inf_{Y>Q_0,\ (Y,S)\in\mathcal{W}^{\mathcal{V}}_{rob,2}} \max_{v\in\mathcal{V}} \inf_{\phi(Y,S,Z,v)\geq 0} tr(Z).$$

With such reformulations, the computation of α_{rob} and β_{rob} become finite-dimensional LMI optimization problems.

2.3 State Estimation Problem

The state estimation problem discussed in this section is the dual of state-feedback control problem.

2.3.1 LQG Optimal Observer for LTV Systems

Here we state linear optimal observer theory of LTV systems, or the well known Kalman filter theory.

Given the LTV system equation as

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(t) & B_{11}(t) & B_{12}(t) & B_2(t) \\ C_2(t) & 0 & I_{n_{d2}} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix} \tag{2.3.1}$$

with the assumption $D_{21}(t) = [0 \quad I_{n_{d2}}]$ holds without lose of generality. $\left[d_1^T(t) \quad d_2^T(t)\right]$ is a white noise process with intensity $\begin{bmatrix} V_{11}(t) & V_{12}(t) \\ V_{12}^T(t) & V_{22}(t) \end{bmatrix}$ for all $t \geq 0$. The initial state $x(0)$ is independent of $d_1(t), d_2(t)$ with $\mathcal{E}\{x(0)\} = \bar{x}_0$, $\mathcal{E}\left\{(x(0)-\bar{x}_0)(x(0)-\bar{x}_0)^T\right\} = Q_0$.

Consider the observer with given input $u(t), t \geq 0$

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B_2(t)u(t) - L(t)\left[y(t) - C_2(t)x(t)\right].$$

We want to find the matrix function $L(\tau), 0 \leq \tau \leq t$, and the initial condition $\hat{x}(0)$, to minimize the criterion

$$\theta := \mathcal{E}\left\{[x(t)-\hat{x}(t)]\,[x(t)-\hat{x}(t)]^T\right\}.$$

The result is given by the following theorem.

Theorem 2.3.1 *Given the LTV system in (2.3.1) with statistics of $x(0)$ and $d(t)$ as above. Then the solution of the optimal observer problem is obtained by choosing for the gain matrix*

$$L^0(t) := -\left[Q(t)C_2^T(t) + \tilde{V}_{12}(t)\right] V_{22}^{-1}(t), \qquad \forall\, t \geq 0,$$

with $Q(t)$ be the solution of the matrix Riccati equation with $Q(0) = Q_0$ and

$$\dot{Q}(t) = \tilde{A}(t)Q(t) + Q(t)\tilde{A}^T(t) - Q(t)C_2^T(t)V_{22}^{-1}(t)C_2(t)Q(t) + \tilde{V}_{11}(t) - \tilde{V}_{12}(t)V_{22}^{-1}(t)\tilde{V}_{12}^T(t)$$

where

$$\begin{aligned}
\tilde{A}(t) &:= A(t) - \tilde{V}_{12}(t)V_{22}^{-1}(t)C_2(t), \\
\tilde{V}_{11}(t) &:= [B_{11}(t) \;\; B_{12}(t)] \begin{bmatrix} V_{11}(t) & V_{12}(t) \\ V_{12}^T(t) & V_{22}(t) \end{bmatrix} \begin{bmatrix} B_{11}^T(t) \\ B_{12}^T(t) \end{bmatrix}, \\
\tilde{V}_{12}(t) &:= B_{11}(t)V_{12}(t) + B_{12}(t)V_{22}(t).
\end{aligned}$$

The initial condition of the observer is chosen to be $\hat{x}(0) = \bar{x}_0$ and

$$\theta = \mathcal{E}\left\{[x(t) - \hat{x}(t)]\,[x(t) - \hat{x}(t)]^T\right\} = Q(t).$$

Proof: The proof is standard and can be found in [KwaS]. ∎

For LPV systems, we know the real-time value of parameter ρ, so it is possible to construct the Kalman filter in real-time. But implementation of Kalman filter relies on solving differential Riccati equation in real-time and increases computation burden, which may not be always suitable and necessary.

2.3.2 Quadratic State Estimator for LPV Systems

Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, and the LPV system

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ C_2(\rho(t)) & 0 & I_{n_{d2}} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix} \tag{2.3.2}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. The statistics of stochastic variables $x(0)$ and $d(t)$ are given by (2.1.3.). We want to synthesize parameter-dependent state estimator in the form of

$$\dot{\hat{x}}(t) = A(\rho(t))\hat{x}(t) + B_2(\rho(t))u(t) - L(\rho(t))\left[y(t) - C_2(\rho(t))\hat{x}(t)\right]$$

where $L(\rho)$ is the state estimation gain.

Similar to LTV case, the criterion we are interested in is given by

$$\theta := \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \mathcal{E}\left\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\right\}.$$

Theorem 2.3.2 *Given a compact set $\mathcal{P}$, and a quadratically detectable LPV system in (2.3.2) and (2.1.3.). Defining the set $\mathcal{Y}_{gs}$ as*

$$\mathcal{Y}_{gs} = \left\{Y \in \mathcal{S}_{+}^{n\times n} : \max_{\rho\in\mathcal{P}} \lambda_{max}\left[Y\tilde{A}(\rho) + \tilde{A}^T(\rho)Y - C_2^T(\rho)V_{22}^{-1}(\rho)C_2(\rho) \right.\right. \\ \left.\left. + Y\left[\tilde{V}_{11}(\rho) - \tilde{V}_{12}(\rho)V_{22}^{-1}(\rho)\tilde{V}_{12}^T(\rho)\right]Y\right] < 0\right\},$$

where

$$\begin{aligned} \tilde{A}(\rho) &:= A(\rho) - \tilde{V}_{12}(\rho)V_{22}^{-1}(\rho)C_2(\rho), \\ \tilde{V}_{11}(\rho) &:= [B_{11}(\rho) \;\; B_{12}(\rho)]\begin{bmatrix} V_{11}(\rho) & V_{12}(\rho) \\ V_{12}^T(\rho) & V_{22}(\rho)\end{bmatrix}\begin{bmatrix} B_{11}^T(\rho) \\ B_{12}^T(\rho)\end{bmatrix}, \\ \tilde{V}_{12}(\rho) &:= B_{11}(\rho)V_{12}(\rho) + B_{12}(\rho)V_{22}(\rho). \end{aligned}$$

Then

$$\theta \leq \inf_{Y \leq Q_0^{-1},\; Y\in\mathcal{Y}_{gs}} tr(Y^{-1}).$$

Proof: For any $Y \in \mathcal{Y}_{gs}$, we can construct a state estimator as

$$\dot{\hat{x}}(t) = A(\rho(t))\hat{x}(t) + B_2(\rho(t))u(t) - L(\rho(t))\left[y(t) - C_2(\rho(t))\hat{x}(t)\right], \tag{2.3.3}$$

where $L(\rho) = -\left[Y^{-1}C_2^T(\rho) + \tilde{V}_{12}(\rho)\right]V_{22}^{-1}(\rho)$. Let $\tilde{x} = x - \hat{x}$, by manipulating equations (2.3.2) and (2.3.3) we get

$$\dot{\tilde{x}}(t) = A_L(\rho(t))\tilde{x}(t) + B_L(\rho(t))\begin{bmatrix} d_1(t) \\ d_2(t)\end{bmatrix}.$$

where $A_L(\rho) = A(\rho) + L(\rho)C_2(\rho)$ and $B_L(\rho) = [B_{11}(\rho) \;\; B_{12}(\rho) + L(\rho)]$. Let $Q(t) := \mathcal{E}\left\{\tilde{x}(t)\tilde{x}^T(t)\right\}$ and $Q(0) = Q_0$, then

$$\dot{Q}(t) = A_L(\rho(t))Q(t) + Q(t)A_L^T(\rho(t)) + B_L(\rho(t)) \begin{bmatrix} V_{11}(\rho(t)) & V_{12}(\rho(t)) \\ V_{12}^T(\rho(t)) & V_{22}(\rho(t)) \end{bmatrix} B_L^T(\rho(t)). \quad (2.3.4)$$

If there exists $Y \in \mathcal{Y}_{gs}$ with $Y \leq Q_0^{-1}$, then there exists a matrix function $\Delta(t)$ such that

$$\begin{aligned} \frac{d(Y^{-1})}{dt} &= 0 \\ &= \tilde{A}(\rho(t))Y^{-1} + Y^{-1}\tilde{A}^T(\rho(t)) - Y^{-1}C_2^T(\rho(t))V_{22}^{-1}(\rho(t))C_2(\rho(t))Y^{-1} \\ &\quad + \tilde{V}_{11}(\rho(t)) - \tilde{V}_{12}(\rho(t))V_{22}^{-1}(\rho(t))\tilde{V}_{12}^T(\rho(t)) + \Delta(t) \\ &= A_L(\rho(t))Y^{-1} + Y^{-1}A_L^T(\rho(t)) + B_L(\rho(t)) \begin{bmatrix} V_{11}(\rho(t)) & V_{12}(\rho(t)) \\ V_{12}^T(\rho(t)) & V_{22}(\rho(t)) \end{bmatrix} B_L^T(\rho(t)) + \Delta(t). \end{aligned} \quad (2.3.5)$$

Subtracting equation (2.3.4) from (2.3.5), we get

$$\frac{d}{dt}\left(Y^{-1} - Q(t)\right) = A_L(\rho(t))\left(Y^{-1} - Q(t)\right) + \left(Y^{-1} - Q(t)\right)A_L^T(\rho(t)) + \Delta(t).$$

So $Y^{-1} \geq Q(t)$ for all t and any trajectory $\rho \in \mathcal{F}_{\mathcal{P}}$. Finally we have

$$\theta \leq \inf_{Y \leq Q_0^{-1},\; Y \in \mathcal{Y}_{gs}} tr(Y^{-1})$$

as expected. ∎

It is also easy to see the similarity between the LTV and LPV state estimation results.

2.4 Quadratic Output-Feedback Control Design

In this section, we study the output-feedback control problem for LPV systems. Motivated by separation principle, we design two full order parameter-dependent output-feedback controllers and derive the bounds explicitly for LQG performance.

2.4.1 LQG Performance of Quadratic Output-Feedback Control

In previous two sections, we studied the quadratic state-feedback control and quadratic state estimation problem for LPV systems. After that, one question is natural to ask: Given an LPV system which does not have complete states available for measurement and feedback, would it be quadratically stabilized by the combination of quadratic state-feedback controller with quadratic state estimator? If this is the case, what is the bound for LQG performance?

The answer for quadratic stabilization is affirmative. The quadratic stability of closed-loop LPV systems using quadratic output-feedback is given in the following theorem.

Theorem 2.4.1 *Given the LPV system $\Sigma_{\mathcal{P}}$ in (2.1.2) and (2.1.3.). If there exist $X \in \mathcal{X}_{gs}$ and $Y \in \mathcal{Y}_{gs}$, where $\mathcal{X}_{gs}$ and $\mathcal{Y}_{gs}$ are defined in Theorem 2.2.2 and Theorem 2.3.2 respectively, then for any $\rho \in \mathcal{F}_{\mathcal{P}}$, the system is quadratically stabilized by the linear parameter-dependent controller $u(t) = F(\rho(t))\hat{x}(t)$. $\hat{x}(t)$ is the state estimator with $\hat{x}(0) = \bar{x}_0$ and*

$$\dot{\hat{x}}(t) = A(\rho(t))\hat{x}(t) + B_2(\rho(t))u(t) - L(\rho(t))\left[y(t) - C_2(\rho(t))\hat{x}(t)\right], \tag{2.4.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$, $F(\rho) = -\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]$ and $L(\rho) = -\left[Y^{-1}C_2^T(\rho) + \tilde{V}_{12}(\rho)\right]V_{22}^{-1}(\rho)$.

Proof: From assumption $Y \in \mathcal{Y}_{gs}$, we have

$$YA_L(\rho) + A_L^T(\rho)Y + YB_L(\rho)\begin{bmatrix} V_{11}(\rho) & V_{12}(\rho) \\ V_{12}^T(\rho) & V_{22}(\rho) \end{bmatrix} B_L^T(\rho)Y < 0, \quad \forall\, \rho \in \mathcal{P}$$

where $A_L(\rho) = A(\rho) + L(\rho)C_2(\rho)$ and $B_L(\rho) = [B_{11}(\rho) \;\; B_{12}(\rho) + L(\rho)]$. So that

$$YA_L(\rho) + A_L^T(\rho)Y < 0$$

for all $\rho \in \mathcal{P}$. This implies that $A_L(\rho)$ is quadratically stable for any parameter trajectory $\rho \in \mathcal{F}_{\mathcal{P}}$. Similarly, the assumption $X \in \mathcal{X}_{gs}$ leads to

$$X^{-1}A_F(\rho) + A_F^T(\rho)X^{-1} < -C_F^T(\rho)C_F(\rho) \le 0, \quad \forall\, \rho \in \mathcal{P}$$

where $A_F(\rho) = A(\rho) + B_2(\rho)F(\rho)$ and $C_F^T(\rho) = \left[C_{11}^T(\rho) \;\; C_{12}^T(\rho) + F^T(\rho)\right]$. So $A_F(\rho)$ is quadratically stable for any trajectory $\rho \in \mathcal{F}_{\mathcal{P}}$.

With the control law $u(t) = F(\rho(t))\hat{x}(t)$ and state estimator defined in (2.4.1), the closed-loop "A" matrix is given by

$$\begin{bmatrix} A(\rho(t)) & B_2(\rho(t))F(\rho(t)) \\ -L(\rho(t))C_2(\rho(t)) & A(\rho(t)) + L(\rho(t))C_2(\rho(t)) + B_2(\rho(t))F(\rho(t)) \end{bmatrix},$$

which can be transformed through a constant similarity matrix into

$$\begin{bmatrix} A_L(\rho(t)) & 0 \\ -L(\rho(t))C_2(\rho(t)) & A_F(\rho(t)) \end{bmatrix}.$$

Since matrices $A_L(\rho)$ and $A_F(\rho)$ are quadratically stable, and the $(2,1)$ block of the above matrix is bounded, the closed-loop system is indeed quadratically stable. ∎

In order to derive LQG performance bound for closed-loop LPV systems, we need some preliminary results about output estimation (OE) and full control (FC) problems for LPV systems. The terms and the relationships between these two problems for LTI systems are explained in [DoyGKF].

Lemma 2.4.1 *Given the LPV full control (FC) system $\Sigma_{\mathcal{P},FC}$ as*

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y_{FC}(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & [I \;\; 0] \\ C_1(\rho(t)) & 0 & 0 & [0 \;\; I] \\ C_2(\rho(t)) & 0 & I & [0 \;\; 0] \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u_{FC}(t) \end{bmatrix}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. The statistics of stochastic variables $x(0)$ and $d_1(t), d_2(t)$ are given by (2.1.3.). The system is quadratically stabilized by controller

$$u_{FC}(t) = K_{\mathcal{P},FC}(\rho(t))\, y_{FC}(t)$$

with $K^T_{\mathcal{P},FC}(\rho) := \begin{bmatrix} L^T(\rho) & 0 \end{bmatrix}$ and $L(\rho) = -\left[Y^{-1}C_2^T(\rho) + \tilde{V}_{12}(\rho)\right]V_{22}^{-1}(\rho)$. Furthermore, for any $Y \in \mathcal{Y}_{gs}$ with $Y < Q_0^{-1}$, the bound for closed-loop LQG performance over finite horizon $[0,T]$ is given by

$$\sigma_T \leq \max_{\rho \in \mathcal{P}} tr\left\{\left[\frac{\delta}{T}\, tr\left(\bar{x}_0\bar{x}_0^T\right) I + Y^{-1}\right] C_1^T(\rho)C_1(\rho)\right\}$$

where $\delta > 0$ is some constant independent of T.

Proof: With the control law $u_{FC}(t)$ given above, the closed-loop system is

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_L(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) + L(\rho(t)) \\ C_1(\rho(t)) & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \end{bmatrix}$$

where $A_L(\rho) = A(\rho) + L(\rho)C_2(\rho)$ and $B_L(\rho) = [B_{11}(\rho) \;\; B_{12}(\rho) + L(\rho)]$. For any $Y \in \mathcal{Y}_{gs}$, we have

$$A_L(\rho)Y^{-1} + Y^{-1}A_L^T(\rho) + B_L(\rho)\begin{bmatrix} V_{11}(\rho) & V_{12}(\rho) \\ V_{12}^T(\rho) & V_{22}(\rho) \end{bmatrix} B_L^T(\rho) < 0$$

for all $\rho \in \mathcal{P}$. It clearly shows quadratical stability of the closed-loop system from Theorem 1.2.1. Similar to the proof of Theorem 1.4.2, we get for any $Y \in \mathcal{Y}_{gs}$ with $Y < Q_0^{-1}$,

$$\sigma_T \leq \max_{\rho\in\mathcal{P}} tr\left\{\left[\frac{\delta}{T}\, tr\left(\bar{x}_0\bar{x}_0^T\right) I + Y^{-1}\right] C_1^T(\rho)C_1(\rho)\right\}$$

with some constant $\delta > 0$ independent of T. ∎

Based on the result for FC case, we can derive the LQG performance bound for LPV OE problem and the result is stated in the following lemma.

Lemma 2.4.2 *Given the LPV output estimation (OE) system $\Sigma_{\mathcal{P},OE}$ as*

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y_{OE}(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & 0 & 0 & I \\ C_2(\rho(t)) & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u_{OE}(t) \end{bmatrix}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$. The stochastic assumptions on $x(0)$ and $d_1(t), d_2(t)$ are defined in (2.1.3.). $A(\rho) - B_2(\rho)C_1(\rho)$ is assumed quadratically stable. Let the controller $K_{\mathcal{P},OE}$ be

$$K_{\mathcal{P},OE} = \mathcal{F}_\ell\left(P_{\mathcal{P},OE}, K_{\mathcal{P},FC}\right)$$

where $K_{\mathcal{P},FC}$ is defined in Lemma 2.4.1 and $P_{\mathcal{P},OE}$ is governed by

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ u_{OE}(t) \\ y_{FC}(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) - B_2(\rho(t))C_1(\rho(t)) & 0 & [-I \;\; B_2(\rho(t))] \\ -C_1(\rho(t)) & 0 & [0 \;\; I] \\ -C_2(\rho(t)) & I & [0 \;\; 0] \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ y_{OE}(t) \\ u_{FC}(t) \end{bmatrix}$$

with $\hat{x}(0) = \bar{x}_0$. Then $K_{\mathcal{P},OE}$ quadratically stabilizes $\Sigma_{\mathcal{P},OE}$. For any $Y \in \mathcal{Y}_{gs}$ with $Y < Q_0^{-1}$, the closed-loop LQG performance over finite horizon $[0,T]$ is bounded by

$$\sigma_T \leq \max_{\rho\in\mathcal{P}} tr\left[Y^{-1}C_1^T(\rho)C_1(\rho)\right].$$

Proof: Wrap the top part of $P_{\mathcal{P},OE}$ with $\Sigma_{\mathcal{P},OE}$ and let $\tilde{x}(t) := x(t) - \hat{x}(t)$, the system can be transformed to

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ e(t) \\ y_{FC}(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & [I \;\; 0] \\ C_1(\rho(t)) & 0 & 0 & [0 \;\; I] \\ C_2(\rho(t)) & 0 & I & [0 \;\; 0] \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ d_1(t) \\ d_2(t) \\ u_{FC}(t) \end{bmatrix}. \tag{2.4.2}$$

This is exactly in the form of FC problem with $\mathcal{E}\{\tilde{x}(0)\} = 0$ and $\mathcal{E}\left\{\tilde{x}(0)\tilde{x}^T(0)\right\} = Q_0$. The LPV system in (2.4.2) is quadratically stabilized by controller $K_{\mathcal{P},FC}$ by Lemma 2.4.1. Note the closed-loop system is

$$\mathcal{F}_\ell(\Sigma_{\mathcal{P},OE}, K_{\mathcal{P},OE}) = \mathcal{F}_\ell(\Sigma_{\mathcal{P},OE}, \mathcal{F}_\ell(P_{\mathcal{P},OE}, K_{\mathcal{P},FC})) = \mathcal{F}_\ell(\mathcal{F}_\ell(\Sigma_{\mathcal{P},OE}, P_{\mathcal{F},OE}), K_{\mathcal{P},FC}),$$

so $K_{\mathcal{P},OE}$ quadratically stabilizes $\Sigma_{\mathcal{P},OE}$. Using Lemma 2.4.1 again, we get

$$\sigma_T \le \max_{\rho\in\mathcal{P}} tr\left[Y^{-1}C_1^T(\rho)C_1(\rho)\right].$$

where $Y \in \mathcal{Y}_{gs}$ and $Y < Q_0^{-1}$. ∎

Remark 2.4.1 *From Lemma 2.4.2, we get the formula of one controller $K_{\mathcal{P},OE}$ for OE problem as*

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ u_{OE}(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) + L(\rho(t))C_2(\rho(t)) - B_2(\rho(t))C_1(\rho(t)) & -L(\rho(t)) \\ -C_1(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ y_{OE}(t) \end{bmatrix}.$$

with initial state $\hat{x}(0) = \bar{x}_0$ and $L(\rho) := -\left[Y^{-1}C_2^T(\rho) + \tilde{V}_{12}(\rho)\right]V_{22}^{-1}(\rho)$.

From Theorem 2.4.1 about the quadratic stability of LPV system using quadratic output-feedback controller and two lemmas about LQG performance of FC and OE problems, we can derive LQG performance bounds explicitly in the following theorem.

Theorem 2.4.2 *Given the LPV system $\Sigma_{\mathcal{P}}$ in (2.1.2) and (2.1.3.), and the control law given by Theorem 2.4.1. If there exists $Y \in \mathcal{Y}_{gs}$ with $Y < Q_0^{-1}$, then the LQG performance σ_∞ of closed-loop system is bounded by*

$$\begin{aligned} \sigma_\infty \;\; &\le \inf_{\substack{X\in\mathcal{X}_{gs} \\ Y<Q_0,\; Y\in\mathcal{Y}_{gs}}} \max_{\rho\in\mathcal{P}} tr\left\{X^{-1}\tilde{V}_{11}(\rho) + Y^{-1}\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]^T\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]\right\} \\ &=: \;\; \gamma. \end{aligned}$$

Proof: Our proof is a mimic of the one for [DoyGKF, Theorem 1]. Defining a new control input $v(t) := u(t) - F(\rho(t))x(t)$, the LPV system $\Sigma_{\mathcal{P}}$ can be transformed to

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} A_F(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ C_{1F}(\rho(t)) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ v(t) \end{bmatrix}.$$

Which means

$$\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \Sigma_{\mathcal{P},c} \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} + U_{\mathcal{P}} v(t) =: \hat{e}(t) + \tilde{e}(t),$$

where $\Sigma_{\mathcal{P},c}$ is governed by

$$\begin{bmatrix} \dot{x}_1(t) \\ \hat{e}(t) \end{bmatrix} = \begin{bmatrix} A_F(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) \\ C_{1F}(\rho(t)) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1(t) \\ d_1(t) \\ d_2(t) \end{bmatrix}$$

with $x_1(0) = x(0)$, and $U_{\mathcal{P}}$ is

$$\begin{bmatrix} \dot{x}_2(t) \\ \tilde{e}(t) \end{bmatrix} = \begin{bmatrix} A_F(\rho(t)) & B_2(\rho(t)) \\ C_{1F}(\rho(t)) & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_2(t) \\ v(t) \end{bmatrix}$$

with $x_2(0) = 0$. Note that $\hat{e}(t)$ is uncorrelated with $\tilde{e}(t)$.

From the assumption $X \in \mathcal{X}_{gs}$, we have

$$A_F(\rho)X + XA_F^T(\rho) + XC_F^T(\rho)C_F(\rho)X < 0 \quad \forall\, \rho \in \mathcal{P}, \tag{2.4.3}$$

so $A_F(\rho)$ is quadratically stable. Following the proof for Theorem 1.4.1 gives

$$\mathcal{E}\left\{\frac{1}{T}\int_0^T \|\hat{e}(t)\|^2 dt\right\} \leq \max_{\rho\in\mathcal{P}} tr\left\{X^{-1}\left[\frac{1}{T}\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + \tilde{V}_{11}(\rho)\right]\right\}$$

for any $X \in \mathcal{X}_{gs}$. Also from equation (2.4.3), there exists some δ less than 1 by continuity, such that

$$X^{-1}A_F(\rho) + A_F^T(\rho)X^{-1} + \frac{1}{\delta^2}C_F^T(\rho)C_F(\rho) + F^T(\rho)\left(1 - \frac{1}{\delta^2}\right)F(\rho) < 0$$

for all $\rho \in \mathcal{P}$, which is equivalent to

$$X^{-1}A_F(\rho) + A_F^T(\rho)X^{-1} + \frac{1}{\delta^2}C_F^T(\rho)C_F(\rho) + \left[X^{-1}B_2(\rho) + \frac{1}{\delta^2}C_F^T(\rho)\begin{bmatrix} 0 \\ I \end{bmatrix}\right]$$

$$\star \left[I - \frac{1}{\delta^2}[0 \;\; I]\begin{bmatrix} 0 \\ I \end{bmatrix}\right]^{-1} \left[B_2^T(\rho)X^{-1} + \frac{1}{\delta^2}[0 \;\; I]\, C_F(\rho)\right] < 0$$

for all $\rho \in \mathcal{P}$. So $U_{\mathcal{P}}$ is a contraction mapping from $\mathbf{L}_2^{n_u} \to \mathbf{L}_2^{n_y}$ for any $\rho \in \mathcal{F}_{\mathcal{P}}$ (see [Bec]), that is

$$\int_0^T \|\tilde{e}(t)\|^2 dt \le \delta^2 \int_0^T \|v(t)\|^2 dt < \int_0^T \|v(t)\|^2 dt.$$

Now, look at system $\Sigma_{\mathcal{P},v}$ which generates signal $v(t)$:

$$\begin{bmatrix} \dot{x}(t) \\ v(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ -F(\rho(t)) & 0 & 0 & I \\ C_2(\rho(t)) & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix}.$$

It is easily to see that an admissible controller $K_{\mathcal{P}}$ quadratically stabilizes original LPV system $\Sigma_{\mathcal{P}}$ if and only if it quadratically stabilizes $\Sigma_{\mathcal{P},v}$. But $\Sigma_{\mathcal{P},v}$ is in the form of OE problem and $A(\rho) + B_2(\rho)F(\rho) = A_F(\rho)$ is quadratically stable. Using Lemma 2.4.2 and Remark 2.4.1, we get the quadratically stabilizing controller $K_{\mathcal{P}}$ for $\Sigma_{\mathcal{P}}$ as

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) + L(\rho(t))C_2(\rho(t)) + B_2(\rho(t))F(\rho(t)) & -L(\rho(t)) \\ F(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ y(t) \end{bmatrix}$$

with $\hat{x}(0) = \bar{x}_0$, which is the same as the state estimator formula in (2.4.1). So we get from Lemma 2.4.2, for any $Y \in \mathcal{Y}_{gs}$ with $Y < Q_0^{-1}$,

$$\mathcal{E}\left\{\frac{1}{T}\int_0^T \|v(t)\|^2 dt\right\} \le \max_{\rho \in \mathcal{P}} tr\left[Y^{-1}F^T(\rho)F(\rho)\right].$$

Then

$$\begin{aligned} \sigma_T &= \mathcal{E}\left\{\frac{1}{T}\int_0^T \left[e_1^T(t)e_1(t) + e_2^T(t)e_2(t)\right] dt\right\} \\ &= \frac{1}{T}\,\mathcal{E}\left\{\int_0^T \|\hat{e}(t)\|^2 dt + \int_0^T \|\tilde{e}(t)\|^2 dt\right\} \qquad (\hat{e}, \tilde{e} \text{ are uncorrelated}) \end{aligned}$$

$$\leq \frac{1}{T}\,\mathcal{E}\left\{\int_0^T \|\hat{e}(t)\|^2 dt + \int_0^T \|v(t)\|^2 dt\right\}$$

$$\leq \max_{\rho\in\mathcal{P}} tr\left\{X^{-1}\left[\frac{1}{T}\left(\bar{x}_0\bar{x}_0^T + Q_0\right) + \tilde{V}_{11}(\rho)\right] + Y^{-1}F^T(\rho)F(\rho)\right\}.$$

Plug in the formula for F and take limit on both sides of above inequality, we finally get

$$\sigma_\infty \leq \inf_{\substack{X\in\mathcal{X}_{gs}\\ Y<Q_0^{-1}, Y\in\mathcal{Y}_{gs}}} \max_{\rho\in\mathcal{P}} tr\left\{X^{-1}\tilde{V}_{11}(\rho) + Y^{-1}\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]^T \left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]\right\}$$

as desired. ∎

2.4.2 LQG Performance of State-Feedback Control plus Kalman Filter

In order to establish the stability and bound LQG performance for LPV systems with output feedback control using Kalman filter, we need to make one additional assumption, which is necessary to guarantee the closed-loop system stability over infinite horizon.

(A5) for any $\rho \in \mathcal{F}_\mathcal{P}$, the pair $[A(\rho(\cdot)),\ B_{11}(\rho(\cdot))]$ is stabilizable as an LTV system (i.e. exponentially stabilized using a bounded state feedback gain $F(t)$, see [RavPK] for details).

It is known from [ShaA1] that the optimal state estimator can be implemented in linear parameter-dependent systems since the observer gains are causal functions of the state-space data. On the contrary, the optimal regulator for LTV system depends anti-causally on the system model, so we are forced to resort to a sub-optimal regulator. The stability of this configuration (quadratic state-feedback + optimal state estimator) is given next:

Theorem 2.4.3 *Given the LPV system $\Sigma_\mathcal{P}$ in (2.1.2) and (2.1.3.). If there exist $X \in \mathcal{X}_{gs}$ and $Y \in \mathcal{Y}_{gs}$, where $\mathcal{X}_{gs}$ and $\mathcal{Y}_{gs}$ are defined in Theorem 2.2.2 and Theorem 2.3.2 respectively, then for all $\rho \in \mathcal{F}_\mathcal{P}$ the system is exponentially stabilized (on infinite horizon) by the linear parameter-dependent controller $u(t) = F(\rho(t))\hat{x}(t)$, where $\hat{x}(t)$ is the state estimation from the Kalman filter defined in Theorem 2.3.2.*

Proof: From the feasibility assumption of $Y \in \mathcal{Y}_{gs}$, we have

$$YA_L(\rho) + A_L^T(\rho)Y + YB_L(\rho)\begin{bmatrix} V_{11}(\rho) & V_{12}(\rho) \\ V_{12}^T(\rho) & V_{22}(\rho)\end{bmatrix} B_L^T(\rho)Y < 0$$

for all $\rho \in \mathcal{P}$. So that

$$Y\left[A-\left(Y^{-1}C_2^T+\tilde{V}_{12}\right)V_{22}^{-1}C_2\right]+\left[A-\left(Y^{-1}C_2^T(\rho)+\tilde{V}_{12}\right)V_{22}^{-1}C_2\right]Y<0$$

for all $\rho \in \mathcal{P}$. This implies that for any trajectory $\rho \in \mathcal{F}_{\mathcal{P}}$, the pair $[A(\rho(\cdot)), C_2(\rho(\cdot))]$ is detectable as an LTV system (see [RavPK]).

The optimal state estimate $\hat{x}(t)$ evolves as $\hat{x}(0) = \bar{x}_0$ and

$$\dot{\hat{x}}(t) = A(\rho(t))\hat{x}(t) + B_2(\rho(t))u(t) - L^0(\rho(t))\left[y(t) - C_2(\rho(t))\hat{x}(t)\right]$$

where $L^0(\rho) := -\left[QC_2^T(\rho) + \tilde{V}_{12}(\rho)\right]V_{22}^{-1}(\rho)$, and $Q(t)$ is the matrix satisfying the Riccati differential equation with $Q(0) = Q_0$ and

$$\begin{aligned}\dot{Q}(t) &= \tilde{A}(\rho(t))Q(t) + Q(t)\tilde{A}^T(\rho(t)) + \tilde{V}_{11}(\rho(t)) - \tilde{V}_{12}(\rho(t))V_{22}^{-1}(\rho(t))\tilde{V}_{12}^T(\rho(t)) \\ &\quad - Q(t)C_2^T(\rho(t))V_{22}^{-1}(\rho(t))C_2(\rho(t))Q(t)\end{aligned}$$

Recall from optimal estimation theory that $\mathcal{E}\left\{(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T\right\} = Q(t)$. The detectability of $[A(\rho(.)), C_2(\rho(.))]$ guarantees existence of bounded matrix $Q(t)$ for all $t \geq 0$. From [RavPK], Assumption $(A5)$ allows us to conclude that the LTV system governed by $\tilde{A}(\rho(t)) - Q(t)C_2^T(\rho(t))V_{22}^{-1}(\rho(t))C_2(\rho(t))$ is exponentially stable.

The remaining part of the proof is similar to the proof of Theorem 2.4.1. ∎

The next lemma is useful in the derivation of the performance bound for LPV systems using output-feedback control with Kalman filter.

Lemma 2.4.3 *Given $V_1(t) \geq 0$, $V_2(t) > 0$ on the interval $[0,T]$. If $Q(t)$ is the solution with $Q(0) = Q_0 > 0$ and*

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t),$$

and there exists $Y \in \mathcal{S}_+^{n\times n}$ with $Y \leq Q_0^{-1}$ satisfying

$$YA(t) + A^T(t)Y - C_2^T(t)V_2^{-1}(t)C_2(t) + YV_1(t)Y < 0,$$

then $Y^{-1} \geq Q(t) > 0$ for all $t \in [0,T]$.

Proof: From the assumption on Y, there exists a time-varying matrix $\tilde{V}_1(t) \geq V_1(t)$ such that

$$\frac{d}{dt}(Y^{-1}) = 0 = A(t)Y^{-1} + Y^{-1}A(t) + \tilde{V}_1(t) - Y^{-1}C^T(t)V_2^{-1}(t)C(t)Y^{-1}.$$

Let $K(\cdot)$ be a bounded matrix, consider two Lyapunov equations on the finite horizon $[0,T]$,

$$\begin{aligned}\dot{Q}_1(t) &= (A(t)+K(t)C(t))\,Q_1(t)+Q_1(t)\,(A(t)+K(t)C(t))^T+V_1(t)+K(t)V_2(t)K(t),\\ \dot{Q}_2(t) &= (A(t)+K(t)C(t))\,Q_2(t)+Q_2(t)\,(A(t)+K(t)C(t))^T+\tilde{V}_1(t)+K(t)V_2(t)K(t)\end{aligned}$$

with initial conditions $Q_1(0)=Q_0$ and $Q_2(0)=Y^{-1}$.

Let $\Phi(t,s)$ be the fundamental matrix associated with $A(t)+K(t)C(t)$. We get the solution of above two Lyapunov equations as follows [Won]:

$$\begin{aligned}Q_1(t) &= \Phi(t,0)Q_0\Phi^T(t,0)+\int_0^T \Phi(t,\tau)\,\left(V_1(\tau)+K(\tau)V_2(\tau)K^T(\tau)\right)\,\Phi^T(t,\tau)\,d\tau,\\ Q_2(t) &= \Phi(t,0)Y^{-1}\Phi^T(t,0)+\int_0^T \Phi(t,\tau)\,\left(\tilde{V}_1(\tau)+K(\tau)V_2(\tau)K^T(\tau)\right)\,\Phi^T(t,\tau)\,d\tau.\end{aligned}$$

It is easy to see that for the same $K(t)$, $Q_2(t)\geq Q_1(t)$ for any $t\in[0,T]$. Let $\tilde{K}(t)=-Y^{-1}C^T(t)V_2^{-1}(t)$, the second Lyapunov equation then yields a constant solution $Q_2(t)\equiv Y^{-1}$, and

$$Y^{-1}\geq Q_1(t)|_{\tilde{K}(t)}. \tag{2.4.4}$$

But the first equation has the optimal solution $Q(t)$ with $\hat{K}(t)=-Q(t)C^T(t)V_2^{-1}(t)$ and generally $\hat{K}(t)\neq\tilde{K}(t)$. From [KwaS, Lemma 3.1], we get

$$Q_1(t)|_{\tilde{K}(t)}\geq Q(t). \tag{2.4.5}$$

Combining equations (2.4.4) with (2.4.5), we get $Y^{-1}\geq Q(t)$ for all $t\in[0,T]$. From the assumptions on V_1 and V_2, it is easy to show from the first Lyapunov equation that $Q(t)>0$ for all $t\in[0,T]$. ∎

Theorem 2.4.4 *Given the LPV system $\Sigma_{\mathcal{P}}$ in (2.1.2) and (2.1.3.). If there exist $X\in\mathcal{X}_{gs}$ and $Y\in\mathcal{Y}_{gs}$ with $Y\leq Q_0^{-1}$, then the exponentially stabilizing control law given in Theorem 2.4.3 yields the closed-loop system performance σ_∞ bounded by*

$$\begin{aligned}\sigma_\infty &\leq \inf_{\substack{X\in\mathcal{X}_{gs}\\ Y\leq Q_0^{-1},Y\in\mathcal{Y}_{gs}}}\ \max_{\rho\in\mathcal{P}} tr\left\{X^{-1}\tilde{V}_{11}+Y^{-1}\left[B_2^T(\rho)X^{-1}+C_{12}(\rho)\right]^T\left[B_2^T(\rho)X^{-1}+C_{12}(\rho)\right]\right\}\\ &= \gamma.\end{aligned}$$

Proof: We have shown in Theorem 2.4.3 that the control law $u(t)=F(\rho(t))\hat{x}(t)$ exponentially stabilizes $\Sigma_{\mathcal{P}}$ for any $\rho\in\mathcal{F}_{\mathcal{P}}$. Using variable transformation $\tilde{u}:=u+C_{12}(\rho)\hat{x}$, the cost criterion becomes

$$\sigma_T=\sup_{\rho\in\mathcal{F}_{\mathcal{P}}}\mathcal{E}\left\{\int_0^T\left[x^T(t)C_{11}^T(\rho(t))C_{11}(\rho(t))x(t)+\tilde{u}^T(t)\tilde{u}(t)\right]dt\right\}+\int_0^T tr\left[C_{12}^T(\rho(t))C_{12}(\rho(t))Q(t)\right]dt.$$

In terms of new control $\tilde{u}$, the observer equation becomes

$$\dot{\hat{x}}(t) = \tilde{A}(\rho(t))\hat{x}(t) + B_2(\rho(t))\tilde{u}(t) - L^0(\rho(t))\left[y(t) - C_2(\rho(t))\hat{x}(t)\right]$$

with $\tilde{A}(\rho) := A(\rho) - B_2(\rho)C_{12}(\rho)$. Also we have

$$\mathcal{E}\left\{x^T(t)C_{11}^T(\rho(t))C_{11}(\rho(t))x(t)\right\} = \mathcal{E}\left\{\hat{x}^T(t)C_{11}^T(\rho(t))C_{11}(\rho(t))\hat{x}(t)\right\} + tr\left[C_{11}^T(\rho(t))C_{11}(\rho(t))Q(t)\right].$$

So the criterion can be rewritten as

$$\begin{aligned}\sigma_T &= \sup_{\rho\in\mathcal{F}_\mathcal{P}} \frac{1}{T}\,\mathcal{E}\left\{\int_0^T \left[\hat{x}^T(t)C_{11}^T(\rho(t))C_{11}(\rho(t))\hat{x}(t) + \tilde{u}^T(t)\tilde{u}(t)\right] dt\right\} \\ &\quad + \sup_{\rho\in\mathcal{F}_\mathcal{P}} \frac{1}{T}\int_0^T tr\left\{\left[C_{11}^T(\rho(t))C_{11}(\rho(t)) + C_{12}^T(\rho(t))C_{12}(\rho(t))\right]Q(t)\right\} dt.\end{aligned}$$

As expected, the first term depends on state estimate $\hat{x}(t)$ and the control, while the last term is independent of the control applied to the system. The first part of the cost is the well-known stochastic linear regulator problem since $y(t) - C_2(\rho(t))\hat{x}(t)$ is a white noise process with intensity $V_{22}(\rho(t))$ (see [KwaS]), and the complete state $\hat{x}(t)$ is available. In our problem formulation, we do not have future knowledge of the time variations in the state-space matrices, hence we cannot use the optimal stochastic linear regulator solution. Instead, we use a sub-optimal linear control law $\tilde{u}(t) = -\hat{B}_2^T(\rho(t))X^{-1}\hat{x}(t)$, which leads to equivalent control $u(t)$ as

$$u(t) = -\left[\hat{B}_2^T(\rho(t))X^{-1} + C_{12}(\rho(t))\right]\hat{x}(t) = F(\rho(t))\hat{x}(t).$$

and $\hat{x}(\cdot)$ is governed by

$$\dot{\hat{x}}(t) = A_F(\rho(t))\hat{x}(t) - L^0(\rho(t))\left[y(t) - C_2(\rho(t))\hat{x}(t)\right].$$

From the assumption $X \in \mathcal{X}_{gs}$, we have

$$A_F(\rho)X + XA_F^T(\rho) + XC_F^T(\rho)C_F(\rho)X < 0$$

for all $\rho \in \mathcal{P}$. From the proof for Theorem 1.4.1 we get

$$\begin{aligned}&\mathcal{E}\left\{\int_0^T \left[\hat{x}^T(t)C_{11}^T(\rho(t))C_{11}(\rho(t))\hat{x}(t) + \tilde{u}^T(t)\tilde{u}(t)\right] dt\right\} \\ &\qquad \le tr\left[X^{-1}\left(\bar{x}_0\bar{x}_0^T + \int_0^T L^0(\rho(t))V_{22}(\rho(t))L^{0T}(\rho(t))\,dt\right)\right],\end{aligned}$$

so that the criterion is bounded by

$$\sigma_T \leq \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \frac{1}{T} tr \left\{ X^{-1} \bar{x}_0 \bar{x}_0^T + \int_0^T X^{-1} \left(Q C_2^T + \tilde{V}_{12} \right) V_{22}^{-1} \left(Q C_2^T + \tilde{V}_{12} \right)^T dt \right.$$
$$\left. + \int_0^T \left(C_{11}^T C_{11} + C_{12}^T C_{12} \right) Q \, dt \right\}.$$

Using the Riccati differential equation of $Q(t)$ and Lemma 2.4.3, we get

$$\sigma_T \leq \sup_{\rho \in \mathcal{F}_{\mathcal{P}}} \frac{1}{T} tr \left\{ X^{-1} \left(\bar{x}_0 \bar{x}_0^T + Q_0 \right) + \int_0^T X^{-1} \tilde{V}_{11}(\rho(t)) dt \right.$$
$$\left. + \int_0^T Y^{-1} \left(B_2^T(\rho(t)) X^{-1} + C_{12}(\rho(t)) \right)^T \left(B_2^T(\rho(t)) X^{-1} + C_{12}(\rho(t)) \right) dt \right\}.$$

Note that the above inequality is true for any $X \in \mathcal{X}_{gs}$ and $Y \in \mathcal{Y}_{gs}$ with $Y < Q_0^{-1}$,

$$\sigma_T \leq \max_{\rho \in \mathcal{P}} tr \left\{ X^{-1} \left[\frac{1}{T} \left(\bar{x}_0 \bar{x}_0^T + Q_0 \right) + \tilde{V}_{11}(\rho) \right] \right.$$
$$\left. + Y^{-1} \left[B_2^T(\rho) X^{-1} + C_{12}(\rho) \right]^T \left[B_2^T(\rho) X^{-1} + C_{12}(\rho) \right] \right\}.$$

Take limit on both sides of above inequality, we have

$$\sigma_\infty \leq \inf_{\substack{X \in \mathcal{X}_{gs} \\ Y \leq Q_0^{-1}, Y \in \mathcal{Y}_{gs}}} \max_{\rho \in \mathcal{P}} tr \left\{ X^{-1} \tilde{V}_{11}(\rho) + Y^{-1} \left[B_2^T(\rho) X^{-1} + C_{12}(\rho) \right]^T \left[B_2^T(\rho) X^{-1} + C_{12}(\rho) \right] \right\}.$$

∎

2.5 Computation of the Bounds and Comments

In this section, we first discuss the general properties of the LQG performance bounds given in Theorem 2.4.2 and Theorem 2.4.4. Then we provide a procedure to compute the bounds. Finally, we give comments on tightness of the bound.

2.5.1 Convexity and Complexity Issues

We should point out that for both output-feedback controllers, the same LQG performance bounds resulted. Now we will figure out how to compute these bounds.

By Schur complement, it is easy to show that the sets $\mathcal{X}_{gs}$, $\mathcal{Y}_{gs}$ define convex constraints for X, Y respectively. So the bound γ is a convex function of X for fixed Y and convex function of Y for fixed X, but is not convex of X, Y jointly. This leads to some difficulty in formulating convex optimization for its computation. Furthermore, the minimization in γ includes infinite number of objectives to be traced, which could be computationally very expensive. So we would like to look for a bound for γ. We propose an "one-step" scheme to compute some bound of γ. This bound is represented as the minimization a function, which is convex of X and Y separately and has infinite convex constraints of parameter ρ. Generally, it can be "minimized" with finite number of constraints by gridding of the compact set $\mathcal{P}$.

Theorem 2.5.1 *Given the compact set $\mathcal{P}$. With the procedure:*

1. *First, do the convex minimization about variable X,*

$$\eta := \inf_{X \in \mathcal{X}_{gs}} tr\ (X^{-1}),$$

pick an $\hat{X}$ which is feasible, and yields a cost close (as close as we want) to η.

2. *Let $W_1 \geq \tilde{V}_{11}(\rho)$, $W_2 \geq \left[B_2^T(\rho)\hat{X}^{-1} + C_{12}(\rho)\right]^T \left[B_2^T(\rho)\hat{X}^{-1} + C_{12}(\rho)\right]$ for all $\rho \in \mathcal{P}$. Next, do the second convex optimization about variable Y,*

$$\omega := \inf_{Y < Q_0^{-1},\ Y \in \mathcal{Y}_{gs}} tr\left(\hat{X}^{-1}W_1 + Y^{-1}W_2\right),$$

similarly, pick feasible $\hat{Y}$ which yields a cost close to ω.

Then the bound for γ is given by

$$\gamma \leq tr\left(\hat{X}^{-1}W_1 + \hat{Y}^{-1}W_2\right) =: \gamma_{sub}.$$

Proof: It is easy to show that η is convex function of X, and ϕ is convex of Y. From assumption, we have $\hat{X} \in \mathcal{X}_{gs}$ and $\hat{Y} \in \mathcal{Y}_{gs}$ with $\hat{Y} < Q_0^{-1}$. So

$$\begin{aligned}
\gamma &= \inf_{\substack{X \in \mathcal{X}_{gs} \\ Y \leq Q_0^{-1}, Y \in \mathcal{Y}_{gs}}} \max_{\rho \in \mathcal{P}} tr\left\{X^{-1}\tilde{V}_{11}(\rho) + Y^{-1}\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]^T\left[B_2^T(\rho)X^{-1} + C_{12}(\rho)\right]\right\} \\
&\leq \inf_{Y \leq Q_0^{-1}, Y \in \mathcal{Y}_{gs}} \max_{\rho \in \mathcal{P}} tr\left\{\hat{X}^{-1}\tilde{V}_{11}(\rho) + Y^{-1}\left[B_2^T(\rho)\hat{X}^{-1} + C_{12}(\rho)\right]^T\left[B_2^T(\rho)\hat{X}^{-1} + C_{12}(\rho)\right]\right\} \\
&\leq tr\left(\hat{X}^{-1}W_1 + \hat{Y}^{-1}W_2\right)
\end{aligned}$$

as desired. ∎

With the standard trick of appending variables, it is easy to convert the minimizations for η and ω in Theorem 2.5.1 to LMI optimization problem.

Under some assumptions on state-space data, the optimization of γ_{sub} will become finite number constraints convex problem. The assumptions are:

- The parameter set $\mathcal{P}$ is convex polytope whose finite set of extreme points is denoted by $\mathcal{V}$,
- $B_2(\rho), C_2(\rho), V_{11}(\rho), V_{12}(\rho)$ and $V_{22}(\rho)$ are constant matrices,
- The parameter dependence of other continuous functions $A(\rho), B_{11}(\rho), B_{12}(\rho), C_{11}(\rho)$ and $C_{12}(\rho)$ are affine on $\rho \in \mathcal{P}$.

The finite-dimensional optimization procedure for computing γ_{sub} is given in the following way:

Theorem 2.5.2 *Given a convex polytope $\mathcal{P}$ with its finite set of extreme points denoted by $\mathcal{V}$, and simplifying assumptions given above. Define finite constraint sets $\mathcal{X}_{gs}^{\mathcal{V}}, \mathcal{Y}_{gs}^{\mathcal{V}}$ as*

$$\begin{aligned}
\mathcal{X}_{gs}^{\mathcal{V}} &:= \Big\{X \in \mathcal{S}_+^{n\times n} : \\
&\quad \max_{v\in\mathcal{V}} \lambda_{max}\Big[\hat{A}(v)X + X\hat{A}^T(v) - B_2B_2^T + XC_{11}^T(v)C_{11}(v)X\Big] < 0\Big\}, \\
\mathcal{Y}_{gs}^{\mathcal{V}} &:= \Big\{Y \in \mathcal{S}_+^{n\times n} : \max_{v\in\mathcal{V}} \lambda_{max}\Big[Y\tilde{A}(v) + \tilde{A}^T(v)Y - C_2^T V_{22}^{-1} C_2 \\
&\quad + Y\Big[\tilde{V}_{11}(v) - \tilde{V}_{12}(v)V_{22}^{-1}\tilde{V}_{12}^T(v)\Big]Y\Big] < 0\Big\}.
\end{aligned}$$

Then we can show $\mathcal{X}_{gs} = \mathcal{X}_{gs}^{\mathcal{V}}$ and $\mathcal{Y}_{gs} = \mathcal{Y}_{gs}^{\mathcal{V}}$, where $\mathcal{X}_{gs}$ and $\mathcal{Y}_{gs}$ are defined in Theorem 2.2.2 and Theorem 2.3.2 respectively. Furthermore, with the same procedure as Theorem 2.5.1 but over finite constraints convex sets $\mathcal{X}_{gs}^{\mathcal{V}}, \mathcal{X}_{gs}^{\mathcal{V}}$.

Proof: The proof is straight forward by itself. ∎

Remark 2.5.1 *Further assume that B_{11}, B_{12} and C_{12} are constant matrices, then we may pick $W_1 = \tilde{V}_{11}$ and $W_2 = \left(B_2\hat{X}^{-1} + C_{12}\right)^T \left(B_2\hat{X}^{-1} + C_{12}\right)$.*

Note this "one-step" scheme of computing bounds γ_{sub}, $\gamma_{sub}^{\mathcal{V}}$ is not guaranteed to find their global minimums. Typically, they only converge to local minimum.

2.5.2 Comments

Here we will study the LQG performance for LTI systems, and discuss its relationship with our bound γ_{sub} in this case. First, let us give the notion of infimum of a set.

Definition 2.5.1 *A subset $\mathcal{W} \subset \mathcal{S}_+^{n\times n}$ is said to have an infimum if there exists a matrix $\hat{W} \in \mathcal{S}_+^{n\times n}$ with the properties:*

- *for any* $W \in \mathcal{W}$, $W \geq \hat{W}$,
- *there exists a sequence of matrices* $\{W_i\}_{i=1}^{\infty} \in \mathcal{W}$, *such that* $W_i \to \hat{W}$.

It is clear that if such a matrix exists, it must be unique. In this case, we denote $\inf \mathcal{W} := \hat{W}$. *Also note that* $\inf_{W \in \mathcal{W}} tr\ (W) = tr\ (\hat{W})$ *and* $\inf_{W \in \mathcal{W}} tr\ (XW) = tr\ (X\hat{W})$ *for any* $X \geq 0$.

The following Lemma is a known fact and listed here for clarity.

Lemma 2.5.1 *Suppose that the pair* (A, B) *is stablizable and the pair* (A, C) *is detectable, then the Algebraic Riccati equation* $PA + A^T P - PBB^T P + C^T C = 0$ *has a maximal solution (see [GohLR]), which we denote* P_+, *and* $A - BB^T P_+$ *is stable. Let the set* $\mathcal{W}$ *be*

$$\mathcal{W} := \left\{ P \in \mathcal{S}_+^{n \times n} : \ \lambda_{max} \left[PA + A^T P - PBB^T P + C^T C \right] < 0 \right\}.$$

Then $\mathcal{W}$ *has an infimum and in fact* $\inf \mathcal{W} = P_+ \geq 0$.

Proof: As (A, B) is stabilizable and (A, C) is detectable, it is easy to show that the maximal solution $P_+ \geq 0$ and $\left(A - BB^T P_+\right)$ is stable. For any feasible $P \in \mathcal{W}$, we define $PA + A^T P + PBB^T P + C^T C := -W < 0$, it equivalent to

$$P\left(A - BB^T P\right) + \left(A - BB^T P\right)^T P = -W - PBB^T P - C^T C < 0 \tag{2.5.1}$$

So $\left(A - BB^T P\right)$ is a stable matrix. Also P_+ satisfies:

$$P_+ A + A^T P_+ - P_+ BB^T P_+ + C^T C = 0,$$

that is

$$P_+ \left(A - BB^T P\right) + \left(A - BB^T P\right)^T P_+ = -P_+ BB^T P - PBB^T P_+ + P_+ BB^T P_+ - C^T C. \tag{2.5.2}$$

Subtracting equation (2.5.2) from (2.5.1) gives us

$$\left(P - P_+\right) A + A^T \left(P - P_+\right) - PBB^T P + P_+ BB^T P_+ = -W,$$

that is

$$\left(P - P_+\right)\left(A - BB^T P\right) + \left(A - BB^T P\right)^T \left(P - P_+\right) = -W - \left(P - P_+\right) BB^T \left(P - P_+\right) < 0.$$

We have shown $\left(A - BB^T P\right)$ is stable, it follows that

$$P > P_+.$$

Also we know from [GohLR] that there exists a sequence $\{P_i\}_{i=1}^{\infty} \in \mathcal{W}$ such that

$$P_iA + A^TP_i - P_iBB^TP_i + C^TC < 0,$$

$P_i \geq P_+$, for $i = 1, 2, \cdots$ and $\lim_{i\to\infty} P_i = P_+$ By definition of the infimum of $\mathcal{W}$, we have $\inf \mathcal{W} = P_+ \geq 0$. ∎

For an LTI system

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_{11} & B_{12} & B_2 \\ C_{11} & 0 & 0 & 0 \\ C_{12} & 0 & 0 & I \\ C_2 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix}.$$

The initial state $x(0)$ is independent of white noise $d_1(t), d_2(t)$ with

$$\begin{aligned} \mathcal{E}\{x(0)\} &:= \bar{x}_0, \\ \mathcal{E}\left\{(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)^T\right\} &:= Q_0, \\ \mathcal{E}\left\{\begin{bmatrix} d_1(t_1) \\ d_2(t_1) \end{bmatrix}\begin{bmatrix} d_1^T(t_2) & d_2^T(t_2) \end{bmatrix}\right\} &:= \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix}\delta(t_1 - t_2) \end{aligned}$$

where $Q_0 > 0$, $V_{11} \geq 0$ and $V_{22} > 0$. With optimal regulator and observer, we know from [KwaS]

$$\sigma_\infty = tr\left[P_+\tilde{V}_{11} + Q_+\left(B_2^TP_+ + C_{12}\right)^T\left(B_2^TP_+ + C_{12}\right)\right].$$

P_+ and Q_+ are the stabilizing positive semi-definite solutions of the following algebraic Riccati equations:

$$\begin{aligned} P_+\hat{A} + \hat{A}^TP_+ - P_+B_2B_2^TP_+ + C_{11}^TC_{11} &= 0, \\ \tilde{A}Q_+ + Q_+\tilde{A}^T - Q_+C_2^TV_{22}^{-1}C_2Q_+ + \tilde{V}_{11} - \tilde{V}_{12}V_{22}^{-1}\tilde{V}_{12}^T &= 0. \end{aligned}$$

where

$$\begin{aligned} \hat{A} &:= A - B_2C_{12}, \quad \tilde{A} := A - \tilde{V}_{12}V_{22}^{-1}C_2, \\ \tilde{V}_{11} &:= [B_{11} \;\; B_{12}]\begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix}\begin{bmatrix} B_{11}^T \\ B_{12}^T \end{bmatrix}, \\ \tilde{V}_{12} &:= B_{11}V_{12} + B_{12}V_{22}. \end{aligned}$$

Furthermore, define

$$\begin{aligned}
\mathcal{W}_1 &:= \left\{P \in \mathcal{S}_+^{n\times n} : \lambda_{max}\left[PA + A^T P - PB_2B_2^T P + C_{11}^T C_{11}\right] < 0\right\}, \\
\hat{P} &:= \inf \mathcal{W}_1, \\
\mathcal{W}_2 &:= \left\{Q \in \mathcal{S}_+^{n\times n} : \lambda_{max}\left[AQ + QA^T - QC_2^T V_{22}^{-1} C_2 Q + \tilde{V}_{11} - \tilde{V}_{12} V_{22}^{-1} \tilde{V}_{12}^T\right] < 0\right\}, \\
\hat{Q} &:= \inf \mathcal{W}_2.
\end{aligned}$$

Note that $\hat{P}, \hat{Q}$ are well-defined, since the sets $\mathcal{W}_1$, $\mathcal{W}_2$ do have infimums in LTI case. From Lemma 2.5.1, $\hat{P} = P_+$ and $\hat{Q} = Q_+$. Then

$$\sigma_\infty = tr\left[\hat{P}\tilde{V}_{11} + \hat{Q}\left(B_2^T\hat{P} + C_{12}\right)^T\left(B_2^T\hat{P} + C_{12}\right)\right].$$

Picking $W_1 = \tilde{V}_{11}$. For arbitrarily small ϵ, our "one-step" scheme given in Theorem 2.5.1 yields $\hat{X}^{-1}$ close to $\hat{P}$, then selecting $W_2 = \left(B_2^T\hat{X}^{-1} + C_{12}\right)^T\left(B_2^T\hat{X}^{-1} + C_{12}\right)$, we get $\hat{Y}^{-1}$ close to $\hat{Q}$ such that

$$\begin{aligned}
\sigma_\infty &\geq tr\left[\hat{X}^{-1}\tilde{V}_{11} + \hat{Y}^{-1}\left(B_2^T\hat{X}^{-1} + C_{12}\right)^T\left(B_2^T\hat{X}^{-1} + C_{12}\right)\right] - \epsilon \\
&= tr\left[\hat{X}^{-1}W_1 + \hat{Y}^{-1}W_2\right] \\
&= \gamma_{sub} - \epsilon.
\end{aligned}$$

This shows that our LQG performance bound γ_{sub} is actually tight in LTI case.

Part II

Induced L_2-Norm Control of LPV Systems

Chapter 3

Analysis of LPV Systems Using Parameter-Dependent Lyapunov Functions

In this chapter we study a class of LPV systems which have bounded parameter variation rates. The previous results in [BecP], [Bec], [ApkG] are based on a single quadratic Lyapunov function approach, and are only suitable for LPV systems with arbitrarily fast parameter variation. By using a parameter-dependent Lyapunov function (PDLF), we formulate an analysis test which is able to exploit the bounded parameter variation information. The test is the generalized Scaled Bounded Real Lemma and leads to a less conservative result for the class of systems we are interested in.

In §3.1, we give a simple example to motivate the usefulness of PDLF for LPV systems. In §3.2 we generalize the quadratic stability notion to parameter-dependent stability of LPV systems by parameter-dependent Lyapunov functions. Finally, in §3.3 we define an induced $\mathbf{L}_2$-norm performance measure for LPV systems with bounded parameter variation rates, and formulate a sufficient condition to test if the induced $\mathbf{L}_2$-norm of an LPV system is less than some $\gamma > 0$.

3.1 Motivation for Using Parameter Dependent Lyapunov Functions

In this section, we will study an example which can not be quadratically stabilized but otherwise is possible by using parameter-dependent Lyapunov function.

Consider the linear parameter-dependent system (modified from [MeyC])

$$\dot{x}(t) = A(\rho(t))x(t) + Bu(t), \tag{3.1.1}$$

where

$$A(\rho) = \begin{bmatrix} a_{11} & a_{12} & \cos(\rho) & \sin(\rho) \\ a_{21} & a_{22} & -\sin(\rho) & \cos(\rho) \\ 0 & 0 & -\tau & 0 \\ 0 & 0 & 0 & -\tau \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \tau & 0 \\ 0 & \tau \end{bmatrix}. \tag{3.1.2}$$

This represents a 2-input system which includes identical actuator dynamics, a time-varying coupling matrix, and 2nd-order plant dynamics.

The quadratic state-feedback stabilization problem is: find a continuous function $F(\rho)$ and a matrix $P \in \mathcal{S}^{4\times 4}, P > 0$ such that

$$[A(\rho) + BF(\rho)]^T P + P[A(\rho) + BF(\rho)] < 0 \tag{3.1.3}$$

for all $\rho \in [-\pi, \pi]$. If such matrices exist, then the parameter-dependent state-feedback law $u(t) = F(\rho(t))x(t)$ would render the closed-loop system exponentially stable for any piecewise continuous trajectory $\rho(\cdot)$.

Unfortunately, if the matrix $A_{11} := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is unstable, then the system (3.1.1) can not be quadratically stabilizable by parameter-dependent state-feedback control. To see it, suppose such matrix $F(\rho)$ exists, then inequality (3.1.3) must hold for both $\rho = 0$ and $\rho = \pi$, that is

$$\begin{aligned} [A(0) + BF(0)]^T P + P[A(0) + BF(0)] &< 0, \\ [A(\pi) + BF(\pi)]^T P + P[A(\pi) + BF(\pi)] &< 0. \end{aligned}$$

Multiplying both equations by 1/2 and adding them up, we get

$$\begin{bmatrix} A_{11} & 0 \\ \star & \star \end{bmatrix}^T P + P \begin{bmatrix} A_{11} & 0 \\ \star & \star \end{bmatrix} < 0.$$

It is clear that no positive definite matrix P satisfies above inequality if A_{11} is unstable [Vid]. So we get a contradiction.

The problem is that there are $\rho(\cdot)$ trajectories which allow the upper $(1,2)$ block of $A(\rho)$ in (3.1.2) to switch between I_2 and $-I_2$ arbitrarily fast. So, regardless of the bandwidth τ of the actuators, the rapidly varying parameter $\rho(t)$ do not allow for quadratic stabilization. Hence, all of the methods in [BecP], [Bec] and [ApkG] are not applicable.

However, a simple singular perturbation argument suggests that the state-feedback

$$F(\rho) := \begin{bmatrix} \begin{bmatrix} \cos(\rho) & -\sin(\rho) \\ \sin(\rho) & \cos(\rho) \end{bmatrix} (-\gamma I_2 - A_{11}) & 0_{2\times 2} \end{bmatrix} \tag{3.1.4}$$

should work, that is, exponentially stabilize LPV systems in equation (3.1.1) with $\rho(\cdot)$ trajectories satisfying

$$\max_{t\geq 0} |\dot{\rho}(t)| \leq B(\tau),$$

where $B(\cdot)$ is some monotonically increasing function of τ. In other words, if there is a known rate bound on $\rho(t)$, then exponentially stabilizing, parameter-dependent state-feedbacks do exist. It is possible to construct a parameter-dependent Lyapunov function which demonstrates the stability of the closed-loop system.

Example 3.1.1 *Given the LPV system in (3.1.1)-(3.1.2) with* $A_{11} = \begin{bmatrix} 0.75 & 2.0 \\ 0 & 0.5 \end{bmatrix}$, *and the state-feedback control law given by (3.1.4). Show the stability of the closed-loop system using PDLF.*

Solution: Note that A_{11} is unstable because of its positive eigenvalues. Define $P(\rho) := P_0 + P_1 \cos(\rho) + P_2 \sin(\rho)$, and PDLF as

$$V(x,\rho) := x^T P(\rho) x.$$

Then the closed-loop system with bounded parameter variation rate $|\dot{\rho}| \leq \nu$ is exponentially stable if $P(\rho(t)) > 0$ and

$$\begin{aligned}\frac{dV}{dt} &= x^T(t)\left\{[A(\rho(t)) + BF(\rho(t))]^T P(\rho(t)) + P(\rho(t))[A(\rho(t)) + BF(\rho(t))] + \frac{dP}{dt}\right\} x(t) \\ &= x^T(t)\left\{[A(\rho(t)) + BF(\rho(t))]^T P(\rho(t)) + P(\rho(t))[A(\rho(t)) + BF(\rho(t))] + \dot{\rho}\frac{dP}{d\rho}\right\} x(t) < 0\end{aligned}$$

for all admissible trajectories $\rho(\cdot)$. Note above equation holds if and only if there exists $P(\rho) > 0$ and

$$[A(\rho) + BF(\rho)]^T P(\rho) + P(\rho)[A(\rho) + BF(\rho)] \pm \nu\frac{dP}{d\rho} < 0 \tag{3.1.5}$$

for all $\rho \in [-\pi, \pi]$. Choose $\nu = 1, \tau = 3.75$ and $\gamma = 0.5$, using feasibility solver (FEASP) in [GahNLC], we solve inequalities (3.1.5) with 20 gridding points by

$$\begin{aligned}P_0 &= \begin{bmatrix} 7319.1 & 6525.7 & -1.7667 & 3.3288 \\ 6525.7 & 8746.1 & 11.729 & -9.3291 \\ -1.7667 & 11.729 & 1058.6 & 3.1193 \\ 3.3288 & -9.3291 & 3.1193 & 1061.2 \end{bmatrix}, \\ P_1 &= \begin{bmatrix} -44.493 & 5.2838 & 1999.4 & 926.67 \\ 5.2838 & 31.857 & 2416.3 & 1354.8 \\ 1999.4 & 2416.3 & 3.9755 & -3.7928 \\ 926.67 & 1354.8 & -3.7928 & -9.5246 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 13.353 & 15.422 & -921.78 & 2001.7 \\ 15.422 & -15.564 & -1346.9 & 2418.1 \\ -921.78 & -1346.9 & 4.5734 & 7.0145 \\ 2001.7 & 2418.1 & 7.0145 & -3.9264 \end{bmatrix}.\end{aligned}$$

Then we check condition (3.1.5) at 360 points (every $1°$). The maximum eigenvalue at these points ranges from -3.04 to -5.95, which clearly indicates that the resulting solution is feasible over the whole parameter interval. So the LPV system with parameter variation $|\dot{\rho}| \leq 1$ is exponentially stable, and the stability is verified by the PDLF defined above. ∎

3.2 Parameter-Dependent Stability of LPV Systems

Before defining parameter-dependent stability for LPV systems using PDLF, we would like to introduce the concept of the parameter ν-variation set.

Definition 3.2.1 Parameter ν-Variation Set
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, finite non-negative numbers $\{\nu_i\}_{i=1}^s$ with $\nu := [\nu_1 \ \cdots \ \nu_s]^T$. We define the parameter ν-variation set as

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \left\{\rho \in C^1(\mathbf{R}, \mathbf{R}^s) : \ \rho(t) \in \mathcal{P}, \ |\dot{\rho}_i| \leq \nu_i, \ i = 1, \cdots, s\right\},$$

where C^1 stands for the class of piecewise continuously differentiable functions.

The LPV systems studied in this part are slightly different because of their state-space data dependence on parameter and its derivative, and they are defined as follows:

Definition 3.2.2 LPV Systems with Bounded Parameter Variation Rates
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, and the continuous functions $(A, B, C, D) : \mathbf{R}^s \times \mathbf{R}^s \to (\mathbf{R}^{n\times n}, \mathbf{R}^{n\times n_d}, \mathbf{R}^{n_e\times n}, \mathbf{R}^{n_e\times n_d})$. An n-th order LPV system with bounded parameter variation rates $\Sigma_{\mathcal{P}}$ is given by

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t), \dot{\rho}(t)) & B(\rho(t), \dot{\rho}(t)) \\ C(\rho(t), \dot{\rho}(t)) & D(\rho(t), \dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}, \tag{3.2.6}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, $x(t) \in \mathbf{R}^n$, $d(t) \in \mathbf{R}^{n_d}$ and $e(t) \in \mathbf{R}^{n_e}$.

Addition to the notations in Definition 1.1.3, we have

- *For $t_0 = 0$, $x(t_0) = 0$, the causal, linear operator $G_\rho : \mathbf{L}_{2,e}^{n_d} \to \mathbf{L}_{2,e}^{n_e}$, is given by*

$$e(t) = \int_{t_0}^{t} C(\rho(t), \dot{\rho}(t))\Phi_\rho(t, \tau)B(\rho(\tau), \dot{\rho}(\tau))d(\tau)d\tau + D(\rho(t), \dot{\rho}(t))d(t),$$

- *The set of causal linear operators described by the LPV system (3.2.6) is denoted as*

$$G_{\mathcal{F}_{\mathcal{P}}^{\nu}} := \{G_\rho : \rho \in \mathcal{F}_{\mathcal{P}}^{\nu}\}.$$

Recall that the definition of quadratic stability involves a single quadratic Lyapunov function. Using PDLF, we can establish the notation of parameter-dependent stability, which is the generalization of quadratic stability concept.

Definition 3.2.3 Parameter-Dependent Stability
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, finite non-negative numbers $\{\nu_i\}_{i=1}^s$, and a function $A \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n\times n})$, the function A is parametrically-dependent stable over $\mathcal{P}$ if there exists a continuously differentiable function $P\colon \mathbf{R}^s \to \mathcal{S}^{n\times n}$ such that, $P(\rho) > 0$ and

$$A^T(\rho,\beta)P(\rho) + P(\rho)A(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P}{\partial \rho_i}\right) < 0 \tag{3.2.7}$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i,\ i = 1, 2, \cdots, s$.

Remark 3.2.1 *If there are no bounds for parameter variation ($\nu_i \to \infty,\ i = 1, \cdots, s$), by restricting P to be a constant matrix, the notation for parameter-dependent stability goes back to quadratic stability. Here we only consider finite $\{\nu_i\}_{i=1}^s$.*

In equation (3.2.7), the left hand side of the inequality is strictly less than zero. Next, we will show that it is actually uniformly negative definite by compactness and continuity.

Lemma 3.2.1 *Given a compact set $\mathcal{P}$, and the LPV system*

$$\dot{x}(t) = A(\rho(t), \dot{\rho}(t))x(t), \tag{3.2.8}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. If the function A is parametrically-dependent stable over $\mathcal{P}$, then there exists some $\delta > 0$, such that

$$A^T(\rho(t), \dot{\rho}(t))P(\rho(t)) + P(\rho(t))A(\rho(t), \dot{\rho}(t)) + \frac{dP}{dt} \le -\delta I_n$$

for all trajectory $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$.

Proof: For compact set $\mathcal{P}$ and finite $\{\nu_i\}_{i=1}^s$, the left hand side of equation (3.2.7) is uniformly negative definite by continuity of function A. So there exists a scalar $\delta > 0$, such that for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i,\ i = 1, 2, \cdots, s$

$$A^T(\rho,\beta)P(\rho) + P(\rho)A(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P}{\partial \rho_i}\right) \le -\delta I_n. \tag{3.2.9}$$

Now, consider any trajectory of $\rho(\cdot)$, which satisfies $\rho(t) \in \mathcal{P}$ and $|\dot{\rho}_i(t)| \le \nu_i,\ i = 1, 2, \cdots, s$ for all t. From equation (3.2.9), we get

$$\begin{aligned} & A^T(\rho(t), \dot{\rho}(t))P(\rho(t)) + P(\rho(t))A(\rho(t), \dot{\rho}(t)) + \sum_{i=1}^{s}\left(\dot{\rho}_i \frac{\partial P}{\partial \rho_i}\right) \\ = \;& A^T(\rho(t), \dot{\rho}(t))P(\rho(t)) + P(\rho(t))A(\rho(t), \dot{\rho}(t)) + \frac{dP}{dt} \\ \le \;& -\delta I_n \end{aligned}$$

holds for any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. ∎

Similar to Lemma 1.2.1 of quadratic stability, the following lemma shows that parameter-dependent stability gives a strong form of robust stability for LPV systems with bounded parameter variation rates.

Lemma 3.2.2 *Given a compact set $\mathcal{P}$, and the LPV system in (3.2.8). If A is parametrically-dependent stable over $\mathcal{P}$, then there exist constant scalars $\gamma_1, \gamma_2 > 0$ such that for any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$,*

$$\|\Phi_\rho(t, t_0)\| \leq \gamma_1 \; e^{[-\gamma_2(t-t_0)]}.$$

Proof: Define parameter-dependent Lyapunov function $V : \mathbf{R}^n \times \mathbf{R}^s \rightarrow \mathbf{R}$ as

$$V(x, \rho) := x^T P(\rho) x,$$

where the function $P(\cdot)$ establishes parameter-dependent stability of A. By the compactness of set $\mathcal{P}$ and continuity of function $P(\rho)$, there exist $\lambda_{\max}, \lambda_{\min} > 0$ which denote respectively the maximum and minimum eigenvalues of $P(\rho)$ over $\mathcal{P}$, such that

$$\lambda_{\min} \|x\|^2 \leq V(x, \rho) \leq \lambda_{\max} \|x\|^2 , \tag{3.2.10}$$

where $x \in \mathbf{R}^n$ and $\rho \in \mathcal{P}$. For any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, the time derivative of $V(x(t), \rho(t))$ along trajectories of LTV system (3.2.8) is

$$\frac{d}{dt} V(x(t), \rho(t)) = x^T(t) \left[A^T(\rho(t), \dot{\rho}(t)) P(\rho(t)) + P(\rho(t)) A(\rho(t), \dot{\rho}(t)) + \frac{dP}{dt} \right] x(t).$$

By Lemma 3.2.1, there exists some $\delta > 0$ such that

$$A^T(\rho(t), \dot{\rho}(t)) P(\rho(t)) + P(\rho(t)) A(\rho(t), \dot{\rho}(t)) + \frac{dP}{dt} \leq -\delta I_n,$$

and

$$\frac{d}{dt} V(x(t), \rho(t)) \leq -\delta \|x(t)\|^2 . \tag{3.2.11}$$

From equation (3.2.10), $-\|x\|^2 \leq \frac{-1}{\lambda_{\max}} V(x, \rho)$. Plug it into equation (3.2.11), we get

$$\frac{d}{dt} V(x(t), \rho(t)) \leq -\frac{\delta}{\lambda_{\max}} V(x(t), \rho(t)).$$

For any $x \neq 0$, $V(x, \rho)$ is non-zero. So above equation can be rewritten as

$$\frac{dV(x(t), \rho(t))}{V(x(t), \rho(t))} \leq -\frac{\delta}{\lambda_{\max}} dt.$$

Then we integrate both sides of this equation from t_0 to t, which leads to

$$V(x(t),\rho(t)) \leq V(x(t_0),\rho(t_0))e^{[-2\gamma_2(t-t_0)]}$$

where $\gamma_2 := \delta/(2\lambda_{\max})$. Using equation (3.2.10) with the above equation, we get

$$\lambda_{\min}\|x(t)\|^2 \leq \lambda_{\max}\|x(t_0)\|^2 e^{[-2\gamma_2(t-t_0)]} \tag{3.2.12}$$

for any $\rho \in \mathcal{F}^{\nu}_{\mathcal{P}}$. Furthermore, the above inequality trivially holds for zero initial condition. Note that $x(t) = \Phi_\rho(t,t_0)x(t_0)$ for all $x(t_0) \in \mathbf{R}^n$. From equation (3.2.12), we finally have

$$\|\Phi_\rho(t,t_0)\| \leq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}e^{[-\gamma_2(t-t_0)]}$$

for all $\rho \in \mathcal{F}_{\mathcal{P}}$. Define $\gamma_1 := (\lambda_{\max}/\lambda_{\min})^{\frac{1}{2}}$, and the result follows. ∎

With the definition of parametrically-dependent stable function in mind, we come up with the parameter-dependent stability concept for LPV systems.

Definition 3.2.4 Parametrically-Dependent Stable LPV System
For an LPV system $\Sigma_{\mathcal{P}}$ in Definition 3.2.2, if function A is parametrically-dependent stable, then $\Sigma_{\mathcal{P}}$ is a parametrically-dependent stable LPV system.

3.3 Induced $\mathbf{L}_2$-Norm Performance and Analysis of LPV systems

In this section we define a performance measure for the LPV system described in Definition 3.2.2 in terms of an induced $\mathbf{L}_2$-norm from the disturbance to error signals, and derive a sufficient condition that guarantees $\Sigma_{\mathcal{P}}$ is parametrically-dependent stable and achieves a prescribed induced $\mathbf{L}_2$-norm performance. This condition generalize the Bounded Real Lemma and is written as an linear matrix inequality (LMI) of some continuously differentiable function.

3.3.1 Induced $\mathbf{L}_2$-Norm Performance Measure

The following lemma establishes the existence of a finite upper bound for the induced $\mathbf{L}_2$-norm over the set of all causal linear operators described by a parametrically-dependent stable LPV system. The proof follows from classical results in linear time-varying (LTV) systems theory [DesV].

Lemma 3.3.1 *Given a parametrically-dependent stable LPV system $\Sigma_{\mathcal{P}}$ in (3.2.6). There exists a finite scalar $M > 0$, such that for zero initial conditions $x(0) := 0$,*

$$\sup_{\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}} \sup_{\|d\|_2 \neq 0,\ d \in \mathbf{L}_2} \frac{\|e\|_2}{\|d\|_2} \leq M < \infty.$$

Proof: For any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$ with initial condition $x(0) = 0$ and input $d \in \mathbf{L}_2$, we have

$$e(t) = \int_0^t C(\rho(t), \dot\rho(t)) \Phi_\rho(t,\tau) B(\rho(\tau), \dot\rho(\tau)) d(\tau) d\tau + D(\rho(t), \dot\rho(t)) d(t).$$

Since functions B, C and D are continuous on the compact set $\mathcal{P}$ and $\{\nu_i\}_{i=1}^s$ are finite numbers, there exist finite scalars $k_B, k_C, k_D > 0$, such that $\|B(\rho,\dot\rho)\| < k_B$, $\|C(\rho,\dot\rho)\| < k_C$ and $\|D(\rho,\dot\rho)\| < k_D$ for all $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. Therefore

$$\begin{aligned}
\|e(t)\| &= \left\| \int_0^t C(\rho(t), \dot\rho(t)) \Phi_\rho(t,\tau) B(\rho(\tau), \dot\rho(\tau)) d(\tau)\ d\tau + D(\rho(t), \dot\rho(t))\ d(t) \right\| \\
&\leq k_B k_C \int_0^t \|\Phi_\rho(t,\tau)\| \|d(\tau)\|\ d\tau + k_D \|d(t)\| \\
&= k_B k_C \int_0^t \|\Phi_\rho(t,\tau)\|^{\frac{1}{2}} \|d(\tau)\| \|\Phi_\rho(t,\tau)\|^{\frac{1}{2}} d\tau + k_D \|d(t)\|.
\end{aligned}$$

Applying Schwarz inequality to the first term of the left hand side, we get

$$\|e(t)\| \leq k_B k_C \left[\int_0^t \|\Phi_\rho(t,\tau)\| \|d(\tau)\|^2 d\tau \right]^{\frac{1}{2}} \left[\int_0^t \|\Phi_\rho(t,\tau)\| d\tau \right]^{\frac{1}{2}} + k_D \|d(t)\|. \qquad (3.3.13)$$

Since $\Sigma_{\mathcal{P}}$ is parametrically-dependent stable, there exists $\gamma_1, \gamma_2 > 0$, such that $\|\Phi_\rho(t,\tau)\| \leq \gamma_1\ e^{[-\gamma_2(t-\tau)]}$ by Lemma 3.2.2. Integrating both sides from $\tau = 0$ to $\tau = t$, we get

$$\int_0^t \|\Phi_\rho(t,\tau)\| d\tau \leq \frac{\gamma_1}{\gamma_2}, \qquad \forall\ t \in [0, \infty).$$

Plug it into equation (3.3.13), then

$$\|e(t)\| \leq k_B k_C \sqrt{\frac{\gamma_1}{\gamma_2}} \left[\int_0^t \|\Phi_\rho(t,\tau)\| \|d(\tau)\|^2 d\tau \right]^{\frac{1}{2}} + k_D \|d(t)\|.$$

Square and integrate both sides on $[0, \infty)$, we get

$$\begin{aligned}
\|e\|_2^2 &\leq \int_0^\infty \left\{ k_B k_C \sqrt{\frac{\gamma_1}{\gamma_2}} \left[\int_0^t \|\Phi_\rho(t,\tau)\| \|d(\tau)\|^2 d\tau \right]^{\frac{1}{2}} + k_D \|d(t)\| \right\}^2 dt \\
&\leq 2 \left\{ k_B^2 k_C^2 \frac{\gamma_1}{\gamma_2} \int_0^\infty \left[\int_0^t \|\Phi_\rho(t,\tau)\| \|d(\tau)\|^2 d\tau \right] dt + k_D^2 \int_0^\infty \|d(\tau)\|^2 d\tau \right\}.
\end{aligned}$$

Exchanging the limits of the first integration, and note $\int_\tau^\infty \|\Phi_\rho(t,\tau)\| dt \leq \gamma_1/\gamma_2$, therefore

$$\begin{aligned}
\|e\|_2^2 &\leq 2 \left\{ k_B^2 k_C^2 \frac{\gamma_1}{\gamma_2} \int_0^\infty \left[\int_\tau^\infty \|\Phi_\rho(t,\tau)\| dt \right] \|d(\tau)\|^2 d\tau + k_D^2 \int_0^\infty \|d(\tau)\|^2 d\tau \right\} \\
&\leq 2 \left(k_B^2 k_C^2 \frac{\gamma_1^2}{\gamma_2^2} + k_D^2 \right) \|d\|_2^2
\end{aligned}$$

holds for all $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. If we define $M := \left[2\left(k_B^2 k_C^2 \gamma_1^2/\gamma_2^2 + k_D^2\right)\right]^{\frac{1}{2}}$, then

$$\sup_{\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}} \sup_{\|d\|_2 \neq 0,\ d \in \mathbf{L}_2} \frac{\|e\|_2}{\|d\|_2} \leq M < \infty.$$

∎

Based on the observation in Lemma 3.3.1, we can define the induced $\mathbf{L}_2$-norm for a parametrically-dependent stable LPV system.

Definition 3.3.1 Induced $\mathbf{L}_2$-norm of Parametrically-Dependent Stable LPV Systems
Given a parametrically-dependent stable LPV system $\Sigma_{\mathcal{P}}$ in (3.2.6), for zero initial conditions $x(0) = 0$, define the induced $\mathbf{L}_2$-norm as

$$\|G_{\mathcal{F}_{\mathcal{P}}^{\nu}}\|_{i,2} \ := \ \sup_{\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}} \sup_{\|d\|_2 \neq 0,\ d \in \mathbf{L}_2} \frac{\|e\|_2}{\|d\|_2}. \qquad (3.3.14)$$

Therefore, the $\mathbf{L}_2$-norm level for an LPV system, represents the largest ratio of disturbance norm to error norm over the set of all causal linear operators described by the LPV system. For a given parametrically-dependent stable LPV system $\Sigma_{\mathcal{P}}$ we denote this norm pictorially as shown in Figure 3.1.

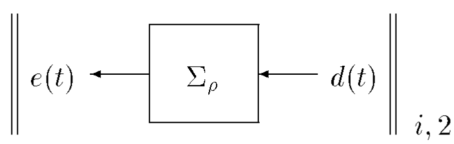

Figure 3.1: Induced $\mathbf{L}_2$-norm for the parametrically-dependent stable LPV system $\Sigma_{\mathcal{P}}$.

3.3.2 Analysis of LPV systems with Induced $\mathbf{L}_2$-Norm Performance

The following fact is useful in the proof of Theorem 3.3.1, and it can be shown similarly to [Bec, Lemma 4.3.2].

Lemma 3.3.2 *Given a parametrically-dependent stable LPV system $\Sigma_{\mathcal{P}}$ in Definition 3.2.2, for $d \in \mathbf{L}_2$ and $x(0) \in \mathbf{R}^n$,*

$$\lim_{t \to \infty} x(t) = 0.$$

Proof: Given $\epsilon > 0$, note that for any $x(0)$,

$$\|x(t)\| = \|\Phi_\rho(t,0)x(0)\| + \left\| \int_0^t \Phi_\rho(t,\tau)B(\rho(\tau),\dot{\rho}(\tau))d(\tau)d\tau \right\|.$$

Using Lemma 3.2.2, there exist t_1, t_2 such that the first term is bounded by $\epsilon/3$ for all $t > t_1$, and the second one is less than $2\epsilon/3$ for all $t > t_2$. Choose $T = \max(t_1, t_2)$, then $\|x(t)\| < \epsilon$ for $t > T$. The proof is done. ∎

Now we give a sufficient condition to check if the induced $\mathbf{L}_2$-norm of an LPV system is less than a prescribed value γ using PDLF.

Theorem 3.3.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, finite non-negative numbers $\{\nu_i\}_{i=1}^s$, and the LPV system in (3.2.6). If there exists a function $W \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ such that, $W(\rho) > 0$ and*

$$\begin{bmatrix} A^T(\rho,\beta)W(\rho) + W(\rho)A(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho)B(\rho,\beta) & \gamma^{-1}C^T(\rho,\beta) \\ B^T(\rho,\beta)W(\rho) & -I_{n_d} & \gamma^{-1}D^T(\rho,\beta) \\ \gamma^{-1}C(\rho,\beta) & \gamma^{-1}D(\rho,\beta) & -I_{n_e} \end{bmatrix} < 0 \quad (3.3.15)$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i, i = 1,\cdots,s$, then

1. *the function A is parametrically-dependent stable over $\mathcal{P}$,*
2. *there exists a scalar δ with $0 \le \delta < \gamma$ such that $\left\|G_{\mathcal{F}_\mathcal{P}^\nu}\right\|_{i,2} \le \delta$.*

Proof: Using the results in [Tad], [RavNK] and [LimAKG], one can show that for trajectories $\rho \in \mathcal{F}_\mathcal{P}^\nu$, the LTV system (3.2.6) is exponentially stable, and the induced $\mathbf{L}_2$-norm from d to e is strictly less than γ. For completeness, we provide an alternative proof here.

For any trajectory $\rho \in \mathcal{F}_\mathcal{P}^\nu$, we have $\rho(t) \in \mathcal{P}$ and $|\dot{\rho}_i(t)| \le \nu_i,\ i = 1,2,\cdots,s$ for all t. Then equation (3.3.15) implies

$$\begin{bmatrix} A^T(\rho,\dot{\rho})W(\rho) + W(\rho)A(\rho,\dot{\rho}) + \sum_{i=1}^{s}\left(\dot{\rho}_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho)B(\rho,\dot{\rho}) & \gamma^{-1}C^T(\rho,\dot{\rho}) \\ B^T(\rho,\dot{\rho})W(\rho) & -I & \gamma^{-1}D^T(\rho,\dot{\rho}) \\ \gamma^{-1}C(\rho,\dot{\rho}) & \gamma^{-1}D(\rho,\dot{\rho}) & -I \end{bmatrix} < 0 \quad (3.3.16)$$

By Schur complement arguments, we get that $\left[I - \gamma^{-2}D^T(\rho,\dot{\rho})D(\rho,\dot{\rho})\right]$ is uniformly negative definite and

$$\begin{aligned} &A^TW + WA + \frac{dW}{dt} + \gamma^{-2}C^TC \\ &\quad + \left(WB + \gamma^{-2}C^TD\right)\left(I - \gamma^{-2}D^TD\right)^{-1}\left(WB + \gamma^{-2}C^TD\right)^T < 0 \end{aligned} \quad (3.3.17)$$

holds for all $t > 0$. Equation (3.3.17) leads to

$$A^T(\rho(t), \dot{\rho}(t))W(\rho(t)) + W(\rho(t))A(\rho(t), \dot{\rho}(t)) + \frac{dW}{dt} < 0$$

for all trajectory $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. So we conclude that function A is parametrically-dependent stable over $\mathcal{P}$.

Consider the PDLF $V: \mathbf{R}^n \times \mathbf{R}^s \to \mathbf{R}$ as

$$V(x, \rho) := x^T W(\rho) x.$$

Along trajectories of (3.2.6), the time derivative of function $V(x(t), \dot{\rho}(t))$ is given by

$$\begin{aligned}
\frac{dV}{dt} &= \left(x^T A^T + d^T B^T\right) W x + x^T W (Ax + Bd) + x^T \frac{dW}{dt} x \\
&= x^T \left(A^T W + WA + \frac{dW}{dt}\right) x + d^T B^T W x + x^T W B d \\
&\leq -x^T \left[\left(WB + \gamma^{-2} C^T D\right)\left(I - \gamma^{-2} D^T D\right)^{-1} \left(WB + \gamma^{-2} C^T D\right)^T + \gamma^{-2} C^T C\right] x \\
&\quad + d^T B^T W x + x^T W B d \\
&\leq -\left\| \left(I - \gamma^{-2} D^T D\right)^{\frac{1}{2}} d - \left(I - \gamma^{-2} D^T D\right)^{-\frac{1}{2}} \left(WB + \gamma^{-2} C^T D\right)^T x \right\|^2 - \gamma^{-2} e^T e + d^T d \\
&\leq -\gamma^{-2} \|e\|^2 + \|d\|^2.
\end{aligned}$$

Integrating both sides from 0 to ∞, starting from $x(0) = 0$. By Lemma 3.3.2, $\lim_{t \to \infty} x(t) = 0$, therefore

$$\|e\|_2^2 \leq \gamma^2 \|d\|_2^2.$$

This indicates that the induced $\mathbf{L}_2$-norm from $d \to e$ is less than or equal to γ. To get the induced $\mathbf{L}_2$-norm of LPV system (3.2.6) strictly less than γ, again by compactness, the inequalities in (3.3.15) can be slightly modified, and still hold. Specifically, there exists a $\delta < \gamma$ such that with B replaced by $(\delta/\gamma)^{-\frac{1}{2}} B$, C replaced by $(\delta/\gamma)^{-\frac{1}{2}} C$ and D replaced by $(\delta/\gamma)^{-1} D$, the inequality (3.3.16) still holds (uniformly). Hence, repeating the arguments above gives that the system described by $\left(A, (\delta/\gamma)^{-\frac{1}{2}} B, (\delta/\gamma)^{-\frac{1}{2}} C, (\delta/\gamma)^{-1} D\right)$ has induced $\mathbf{L}_2$-norm less than γ, reaching the desired conclusion. ∎

Remark 3.3.1 *In Theorem 3.3.1, the parameter variation rates are assumed in symmetric region about zero, but it is possible to relax such assumption to non-symmetric case. The analysis and synthesis results given in Theorem 3.3.1 and Theorem 4.3.1 (or Theorem 4.3.2) still hold with minor modification.*

Theorem 3.3.1 is the generalization of well known Scaled Bounded Real Lemma. and formulates a sufficient condition to test the induced $\mathbf{L}_2$-norm, from disturbance to error signals, of an LPV system is less than some given performance level $\gamma > 0$. By restricting constant matrix W, it recovers the result in [Bec, Lemma 3.4.5] which is a analysis test for LPV systems with arbitrarily fast varying parameter. Our analysis test (Theorem 3.3.1) is potentially more powerful and less conservative than previous results because of its exploitation of parameter variation rates information. Moreover, the condition (3.3.15) is written as a group of LMIs of a continuously differentiable function $W(\rho)$, so it is a infinite dimensional convex problem. By approximating the function space with finite basis functions, we can simplify the condition to finite dimensional convex problem and solve the analysis problem using efficient convex optimization techniques.

If the state-space data do not depend on derivative of parameter explicitly, we have the following simplified analysis test. This test includes 2^s LMIs of some positive definite function, and only needs to grid parameter space thus computationally less expensive than Theorem 3.3.1.

Corollary 3.3.1 *Given the LPV system in (3.2.6) without state-space data dependence on parameter derivative. If there exists a function* $W \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *such that,* $W(\rho) > 0$ *and*

$$\begin{bmatrix} A^T(\rho)W(\rho) + W(\rho)A(\rho) + \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho)B(\rho) & \gamma^{-1}C^T(\rho) \\ B^T(\rho)W(\rho) & -I_{n_d} & \gamma^{-1}D^T(\rho) \\ \gamma^{-1}C(\rho) & \gamma^{-1}D(\rho) & -I_{n_e} \end{bmatrix} < 0 \quad (3.3.18)$$

for all $\rho \in \mathcal{P}$*, then*

1. *the function* A *is parametrically-dependent stable over* $\mathcal{P}$*,*
2. *there exists a scalar* δ *with* $0 \le \delta < \gamma$ *such that* $\left\| G_{\mathcal{F}_{\mathcal{P}}^{\nu}} \right\|_{i,2} \le \delta$*.*

Proof: It follows similarly to the one for Theorem 3.3.1. ∎

Remark 3.3.2 *The notation* $\sum_{i=1}^{s} \pm(\cdot)$ *in (3.3.18) indicates that every combination of* $+(\cdot)$ *and* $-(\cdot)$ *should be included in the inequality. This means that the* 3×3 *"inequality" actually represents* 2^s *different inequalities which must be checked simultaneously.*

As is customary [Sch2], [Bec], [BecP], [Pac], [ApkG], [ApkGB], we will use analysis result Theorem 3.3.1 to derive the existence condition for control synthesis.

Chapter 4

Control of LPV Systems with Induced $\mathbf{L}_2$-Norm Performance

In this chapter, we study a parameter-dependent output-feedback control problem for LPV systems with bounded parameter variation rates. This problem determines the existence of a parameter-dependent controller to parametrically-dependent stabilize the closed-loop LPV system and guarantee the induced $\mathbf{L}_2$-norm of the closed-loop system less than γ. The derivative of parameter is assumed to be measurable in real-time to construct such a controller.

In §4.1, we define the Parameter-Dependent γ-Performance Problem for LPV systems, which is generalized LPV version of the standard $\mathcal{H}_\infty$ problem [Fra]. In §4.2, we solve for the parameter-dependent state-feedback control problem. This problem has its own interest other than being used to derive the solution for the Parameter-Dependent γ-Performance Problem. In §4.3, we derive the necessary and sufficient condition for the Parameter-Dependent γ-Performance Problem. Finally, we study computational issues related to the Parameter-Dependent γ-Performance Problem in §4.4.

4.1 Parameter-Dependent γ-Performance Problem

In this section we define the Parameter-Dependent γ-Performance Problem for LPV systems. The definition is the application of analysis result in Theorem 3.3.1 to the closed-loop systems.

The open-loop LPV systems are in the standard form and given as follows.

Definition 4.1.1 Open-Loop LPV systems for Induced L_2-Norm Control
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, consider the open-loop LPV system $\Sigma_{\mathcal{P}}$

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & D_{11}(\rho) & D_{12}(\rho(t)) \\ C_2(\rho(t)) & D_{21}(\rho(t)) & D_{22}(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix} \tag{4.1.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, $x(t) \in \mathbf{R}^n$, $d(t) \in \mathbf{R}^{n_d}$, $e(t) \in \mathbf{R}^{n_e}$, $u(t) \in \mathbf{R}^{n_u}$ and $y(t) \in \mathbf{R}^{n_y}$. All of the state-space matrices are of appropriate dimensions.

For simplification, we made the following assumptions for the generalized plant:

(B1) $D_{22}(\rho) = 0_{n_y \times n_u}$,

(B2) $D_{12}(\rho)$ is full column rank for all $\rho \in \mathcal{P}$,

(B3) $D_{21}(\rho)$ is full row rank for all $\rho \in \mathcal{P}$,

Assumption $(B1)$ can be relaxed easily by including a feed through term to the controller for the modified plant which has D_{22} term equal to zero. The relaxation of Assumptions $(B2)$ and $(B3)$ leads to singular $\mathcal{H}_\infty$ problem [Sch1], [Sch2]. With the Assumptions $(B1) - (B3)$, we can simplify the open-loop LPV systems using techniques described in [Bec], which lead to:

Definition 4.1.2 Simplified Open-Loop LPV Systems for Induced L_2-norm Control
Given the open-loop LPV system $\Sigma_{\mathcal{P}}$ in Definition 4.1.1 and the Assumptions $(B1) - (B3)$ hold, then the system can be rewritten as

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ C_{11}(\rho(t)) & D_{1111}(\rho(t)) & D_{1112}(\rho(t)) & 0 \\ C_{12}(\rho(t)) & D_{1121}(\rho(t)) & D_{1122}(\rho(t)) & I_{n_{e2}} \\ C_2(\rho(t)) & 0 & I_{n_{d2}} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix} \tag{4.1.2}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, $d_1(t) \in \mathbf{R}^{n_{d1}}$, $d_2(t) \in \mathbf{R}^{n_{d2}}$, $e_1(t) \in \mathbf{R}^{n_{e1}}$ and $e_2(t) \in \mathbf{R}^{n_{e2}}$.

The class of finite dimensional parameter-dependent controllers, which depend on parameters and their derivatives, is given by

Definition 4.1.3 Parameter-Dependent Controllers

Given a compact set $\mathcal{P} \subset \mathbf{R}^s$ and an integer $m \geq 0$, denote the parametrically-dependent, m-dimensional linear feedback controller as $K_{\mathcal{P}}$, with the continuous functions (A_K, B_K, C_K, D_K) : $\mathbf{R}^s \times \mathbf{R}^s \to (\mathbf{R}^{m\times m}, \mathbf{R}^{m\times n_y}, \mathbf{R}^{n_u\times m}, \mathbf{R}^{n_u\times n_y})$. The controller $K_{\mathcal{P}}$ depends on parameter and its derivative, and is written as

$$\begin{bmatrix} \dot{x}_k(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_K(\rho(t),\dot{\rho}(t)) & B_K(\rho(t),\dot{\rho}(t)) \\ C_K(\rho(t),\dot{\rho}(t)) & D_K(\rho(t),\dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x_k(t) \\ y(t) \end{bmatrix} \tag{4.1.3}$$

where $\rho \in \mathcal{F}^{\nu}_{\mathcal{P}}$, $x_k(t)$ is the m-dimensional controller states.

Define $x^T_{\text{clp}}(t) := \begin{bmatrix} x^T(t) & x^T_k(t) \end{bmatrix}$, $e^T(t) := \begin{bmatrix} e^T_1(t) & e^T_2(t) \end{bmatrix}$ and $d^T(t) := \begin{bmatrix} d^T_1(t) & d^T_2(t) \end{bmatrix}$. Then the closed-loop LPV system is given by

$$\begin{bmatrix} \dot{x}_{\text{clp}}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{\text{clp}}(\rho(t),\dot{\rho}(t)) & B_{\text{clp}}(\rho(t),\dot{\rho}(t)) \\ C_{\text{clp}}(\rho(t),\dot{\rho}(t)) & D_{\text{clp}}(\rho(t),\dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x_{\text{clp}}(t) \\ d(t) \end{bmatrix},$$

where

$$A_{\text{clp}}(\rho,\dot{\rho}) := \begin{bmatrix} A(\rho) + B_2(\rho)D_K(\rho,\dot{\rho})C_2(\rho) & B_2(\rho)C_K(\rho,\dot{\rho}) \\ B_K(\rho,\dot{\rho})C_2(\rho) & A_K(\rho,\dot{\rho}) \end{bmatrix}, \tag{4.1.4.a}$$

$$B_{\text{clp}}(\rho,\dot{\rho}) := \begin{bmatrix} B_{11}(\rho) & B_{12}(\rho) + B_2(\rho)D_K(\rho,\dot{\rho}) \\ 0 & B_K(\rho,\dot{\rho}) \end{bmatrix}, \tag{4.1.4.b}$$

$$C_{\text{clp}}(\rho,\dot{\rho}) := \begin{bmatrix} C_{11}(\rho) & 0 \\ C_{12}(\rho) + D_K(\rho,\dot{\rho})C_2(\rho) & C_K(\rho,\dot{\rho}) \end{bmatrix}, \tag{4.1.4.c}$$

$$D_{\text{clp}}(\rho,\dot{\rho}) := \begin{bmatrix} D_{1111}(\rho) & D_{1112}(\rho) \\ D_{1121}(\rho) & D_{1122}(\rho) + D_K(\rho,\dot{\rho}) \end{bmatrix}. \tag{4.1.4.d}$$

Next we will define the Parameter-Dependent γ-Performance Problem. Given a parameter-dependent plant, the Parameter-Dependent γ-Performance Problem is to determine if there exists a parameter-dependent controller and a PDLF such that the analysis test described in Theorem 3.3.1 holds for the closed-loop system.

Definition 4.1.4 Parameter-Dependent γ-Performance Problem

Given the open-loop LPV system $\Sigma_{\mathcal{P}}$ in Definition 4.1.2, and the performance level $\gamma >$

0. *The Parameter-Dependent γ-Performance Problem is solvable if there exist an integer $m \geq 0$, a function $W \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{(n+m)\times(n+m)})$, and continuous matrix functions $(A_K, B_K, C_K, D_K) : \mathbf{R}^s \times \mathbf{R}^s \to (\mathbf{R}^{m\times m}, \mathbf{R}^{m\times n_y}, \mathbf{R}^{n_u\times m}, \mathbf{R}^{n_u\times n_y})$ such that $W(\rho) > 0$ and*

$$\begin{bmatrix} A_{\mathrm{clp}}^T(\rho,\beta)W(\rho) + W(\rho)A_{\mathrm{clp}}(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho)B_{\mathrm{clp}}(\rho,\beta) & \gamma^{-1}C_{\mathrm{clp}}^T(\rho,\beta) \\ B_{\mathrm{clp}}^T(\rho,\beta)W(\rho) & -I_{n_d} & \gamma^{-1}D_{\mathrm{clp}}^T(\rho,\beta) \\ \gamma^{-1}C_{\mathrm{clp}}(\rho,\beta) & \gamma^{-1}D_{\mathrm{clp}}(\rho,\beta) & -I_{n_e} \end{bmatrix} < 0 \tag{4.1.5}$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i$, $i = 1, \cdots, s$. Here the matrices $A_{\mathrm{clp}}, B_{\mathrm{clp}}, C_{\mathrm{clp}}$ and D_{clp} are defined in equation (4.1.4.).

This problem is a generalization of the standard sub-optimal $\mathcal{H}_\infty$ optimal control problem, and conceptually expands the applicability and usefulness of the $\mathcal{H}_\infty$ control methodology. Additionally, the solution can be put inside a larger design iteration, such as a $D - K$ iteration, to achieve robustness to other perturbations, such as unmodeled dynamics.

Before solving the Parameter-Dependent γ-Performance Problem, we will study the state-feedback control problem for LPV systems in the next section.

4.2 Parameter-Dependent State-Feedback Problem

In this section we study the Parameter-Dependent State-Feedback Problem. The problem is about the existence of parameter-dependent state-feedback control to stabilize the closed-loop system and make the induced $\mathbf{L}_2$-norm less than a specified performance level γ. First, we define the problem we want to solve.

Definition 4.2.1 Parameter-Dependent State-Feedback Problem
Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, the performance level $\gamma > 0$, and the LPV system $\Sigma_{\mathcal{P}}$

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & 0 & D_{12}(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix} \tag{4.2.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. *The Parameter-Dependent State-Feedback Problem is solvable if there exist functions* $Z \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $F \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n_u \times n})$, *such that for all* $\rho \in \mathcal{P}$ *and* $|\beta_i| \leq \nu_i, i = 1, 2, \cdots, s$, $Z(\rho) > 0$ *and*

$$\begin{bmatrix} A_F^T(\rho,\beta)Z(\rho) + Z(\rho)A_F(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial Z}{\partial \rho_i}\right) & Z(\rho)B_1(\rho) & \gamma^{-1}C_F^T(\rho,\beta) \\ B_1^T(\rho)Z(\rho) & -I_{n_d} & 0 \\ \gamma^{-1}C_F(\rho,\beta) & 0 & -I_{n_e} \end{bmatrix} < 0, \quad (4.2.2)$$

where $A_F(\rho,\dot{\rho}) := A(\rho) + B_2(\rho)F(\rho,\dot{\rho})$, $C_F(\rho,\dot{\rho}) := C_1(\rho) + D_{12}(\rho)F(\rho,\dot{\rho})$.

With Assumption (B2), the open-loop LPV system for the state-feedback problem can be written as

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_{11}(\rho(t)) & 0 & 0 \\ C_{12}(\rho(t)) & 0 & I_{n_{e2}} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix} \quad (4.2.3)$$

If the Parameter-Dependent State-Feedback Problem is solvable, then the state-feedback control law $u = F(\rho,\dot{\rho})x$ would render the closed-loop system exponentially stable and induced $\mathbf{L}_2$-norm less than γ. The following theorem states the existence condition of a state-feedback controller for Parameter-Dependent State-Feedback Problem in the form of LMIs expressed by the state-space data of the open-loop LPV systems.

Theorem 4.2.1 *Given a compact set* $\mathcal{P} \subset \mathbf{R}^s$, *the performance level* $\gamma > 0$, *and the LPV system in (4.2.3). The Parameter-Dependent State-Feedback Problem is solvable if and only if there exists a function* $X \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *such that for all* $\rho \in \mathcal{P}$, $X(\rho) > 0$ *and*

$$\begin{bmatrix} X(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)X(\rho) - \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial X}{\partial \rho_i}\right) - B_2(\rho)B_2^T(\rho) & X(\rho)C_{11}^T(\rho) & \gamma^{-1}B_1(\rho) \\ C_{11}(\rho)X(\rho) & -I_{e_1} & 0 \\ \gamma^{-1}B_1^T(\rho) & 0 & -I_d \end{bmatrix} < 0 \quad (4.2.4)$$

where $\hat{A}(\rho) := A(\rho) - B_2(\rho)C_{12}(\rho)$.

Proof: $\Rightarrow$ Define matrix functions $R(\rho)$, $U(\rho)$ and $V(\rho)$ as follows.

$$R(\rho,\beta) \quad := \quad \begin{bmatrix} A^T(\rho)Z(\rho)+Z(\rho)A(\rho)+\sum_{i=1}^{s}\left(\beta_i\frac{\partial Z}{\partial\rho_i}\right) & Z(\rho)B_1(\rho) & \gamma^{-1}C_{11}^T(\rho) & \gamma^{-1}C_{12}^T(\rho) \\ B_1^T(\rho)Z(\rho) & -I_d & 0 & 0 \\ \gamma^{-1}C_{11}(\rho) & 0 & -I_{n_{e1}} & 0 \\ \gamma^{-1}C_{12}(\rho) & 0 & 0 & -I_{n_{e2}} \end{bmatrix},$$

$$U(\rho) \quad := \quad \begin{bmatrix} Z(\rho)B_2(\rho) \\ 0 \\ 0 \\ \gamma^{-1}I_{n_u} \end{bmatrix}, \quad V^T(\rho) := [I_n \quad 0 \quad 0 \quad 0].$$

Then equation (4.2.2) can be rewritten with these new notations as

$$G(\rho,\beta) := R(\rho,\beta) + U(\rho)F(\rho,\beta)V^T(\rho) + V(\rho)F^T(\rho,\beta)U^T(\rho) < 0.$$

Let $X(\rho) := \gamma^{-2}Z^{-1}(\rho)$. The orthonormal bases of $U(\rho)$ and $V(\rho)$ are given by

$$U_\perp(\rho) \quad := \quad \begin{bmatrix} \gamma X(\rho) & 0 & 0 \\ 0 & I_{n_d} & 0 \\ 0 & 0 & I_{n_{e1}} \\ -B_2^T(\rho) & 0 & 0 \end{bmatrix}, \quad V_\perp(\rho) := \begin{bmatrix} 0 & 0 & 0 \\ I_{n_d} & 0 & 0 \\ 0 & I_{n_{e1}} & 0 \\ 0 & 0 & I_{n_{e2}} \end{bmatrix}.$$

Since $G(\rho,\beta)$ is uniformly negative definite over the compact set $\mathcal{P}$ and $|\beta_i| \le \nu_i, i = 1,2,\cdots,s$, and $U_\perp(\rho), V_\perp(\rho)$ are of full column rank for all $\rho \in \mathcal{P}$, it is clear that if $G(\rho,\beta) < 0$ for all $\rho \in \mathcal{P}$, then

$$U_\perp^T(\rho)G(\rho,\beta)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp^T(\rho)G(\rho,\beta)V_\perp(\rho) < 0,$$

which implies that

$$U_\perp^T(\rho)R(\rho,\beta)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp^T(\rho)R(\rho,\beta)V_\perp(\rho) < 0$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i, i = 1,2,\cdots,s$. But $V_\perp^T(\rho)R(\rho,\beta)V_\perp(\rho) < 0$ yields no useful ingotmation, and the inequality $U_\perp^T(\rho)R(\rho,\beta)U_\perp(\rho) < 0$ is identical to equation (4.2.4) by simple algebra.

$\Leftarrow$ For sufficiency, we need to show that equation (4.2.4) establishes a matrix function $F(\rho,\beta)$ such that equation (4.2.2) with such a F holds for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i, i =$

$1, 2, \cdots, s$. By Schur complement, equation (4.2.4) can be written as

$$X(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)X(\rho) - \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial X}{\partial \rho_i}\right) - B_2(\rho)B_2^T(\rho) + \gamma^{-2}B_1(\rho)B_1^T(\rho)$$
$$+ X(\rho)C_{11}^T(\rho)C_{11}(\rho)X(\rho) < 0 \qquad (4.2.5)$$

for all $\rho \in \mathcal{P}$. Define $Z(\rho) := \gamma^{-2}X^{-1}(\rho)$, then

$$\frac{\partial Z}{\partial \rho_i} = -\gamma^{-2}X^{-1}\frac{\partial X}{\partial \rho_i}X^{-1}$$

for $i = 1, \cdots, s$. Pre and post-multiply the left hand side of equation (4.2.5) by $Z(\rho)$, and factor out a γ^{-2} results in

$$\hat{A}^T(\rho)Z(\rho) + Z(\rho)\hat{A}(\rho) + \gamma^{-2}\sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial X^{-1}}{\partial \rho_i}\right) - \gamma^2 Z(\rho)B_2(\rho)B_2^T(\rho)Z(\rho)$$
$$+ Z(\rho)B_1(\rho)B_1^T(\rho)Z(\rho) + \gamma^{-2}C_{11}^T(\rho)C_{11}(\rho) < 0, \quad \forall\, \rho \in \mathcal{P}$$

which is equivalent to

$$\left[A - B_2\left(C_{12} + \gamma^2 B_2^T Z\right)\right]^T Z + Z\left[A - B_2\left(C_{12} + \gamma^2 B_2^T Z\right)\right] + \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial Z}{\partial \rho_i}\right)$$
$$+\gamma^{-2}\left[C_{11}^T \;\; C_{12}^T - \left(C_{12}^T + \gamma^2 Z B_2\right)\right]\begin{bmatrix} C_{11} \\ C_{12} - \left(C_{12} + \gamma^2 B_2^T Z\right) \end{bmatrix} + ZB_1B_1^T Z < 0. \qquad (4.2.6)$$

for all $\rho \in \mathcal{P}$. Equation (4.2.6) results in a natural choice of state-feedback gain F

$$F(\rho) := -\left[\gamma^2 B_2^T(\rho)Z(\rho) + C_{12}(\rho)\right] = -\left[B_2^T(\rho)X^{-1}(\rho) + C_{12}(\rho)\right], \qquad (4.2.7)$$

such that

$$A_F^T(\rho)Z(\rho) + Z(\rho)A_F(\rho) + \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial Z}{\partial \rho_i}\right) + \gamma^{-2}C_F^T(\rho)C_F(\rho) + Z(\rho)B_1(\rho)B_1^T(\rho)Z(\rho) < 0$$

for all $\rho \in \mathcal{P}$. So for any $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i,\; i = 1, \cdots, s$, we have

$$A_F^T(\rho)Z(\rho) + Z(\rho)A_F(\rho) + \sum_{i=1}^{s} \left(\beta_i \frac{\partial Z}{\partial \rho_i}\right) + \gamma^{-2}C_F^T(\rho)C_F(\rho) + Z(\rho)B_1(\rho)B_1^T(\rho)Z(\rho) < 0.$$

Note that the above inequality is exactly the Schur complement of equation (4.2.2). ∎

Theorem 4.2.1 converts the solvability condition of Parameter-Dependent State-Feedback Problem to existence of one matrix function $X(\rho)$ satisfying condition (4.2.4) (which involves 2^s inequalities). The variable X is shown in affine form in all of the inequalities. This LMI formulation have advantage over equation (4.2.2) from computational point of view. Furthermore, the theorem says that even if we search the state-feedback controllers which may depend on parameter and its derivative for Parameter-Dependent State-Feedback Problem, it is enough to render the closed-loop system's induced $\mathbf{L}_2$-norm performance less than γ by state-feedback controller in the form of equation (4.2.7), which depend on parameter only. This theorem will be used later to derive the existence conditions for the Parameter-Dependent γ-Performance Problem.

4.3 Parameter-Dependent Output-Feedback Controller Synthesis

In this section, we derive the solvability conditions for the Parameter-Dependent γ-Performance Problem. The synthesis result exploits the parameter variation information by using PDLF, thus is less conservative than single quadratic Lyapunov function approach [BecP], [Bec], [ApkG].

First, we will study the case of $D_{11}(\rho) = 0$, in which the formula is much simpler. Then we generalize the result to $D_{11}(\rho) \neq 0$ case.

4.3.1 $D_{11}(\rho) = 0$ case

We state a matrix fact which will be used later in the proof of our synthesis result.

Lemma 4.3.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, and two functions $X \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$, $Y \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ with $X(\rho) > 0$, $Y(\rho) > 0$ for all $\rho \in \mathcal{P}$, a positive integer m. There exist matrix functions $X_2 \in \mathcal{C}^1(\mathbf{R}^s, \mathbf{R}^{n\times m})$, $X_3 \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{m\times m})$ such that*

$$\begin{bmatrix} X(\rho) & X_2(\rho) \\ X_2^T(\rho) & X_3(\rho) \end{bmatrix} > 0,$$

and

$$\begin{bmatrix} X(\rho) & X_2(\rho) \\ X_2^T(\rho) & X_3(\rho) \end{bmatrix}^{-1} = \begin{bmatrix} Y(\rho) & \star \\ \star & \star \end{bmatrix}, \qquad (\text{“}\star\text{” means “don't care”})$$

for all $\rho \in \mathcal{P}$ if and only if

$$\begin{bmatrix} X(\rho) & I_n \\ I_n & Y(\rho) \end{bmatrix} \geq 0, \quad \text{and} \quad \text{rank}\left(\begin{bmatrix} X(\rho) & I_n \\ I_n & Y(\rho) \end{bmatrix}\right) \leq (n+m).$$

Furthermore, we have

$$\frac{\partial Y}{\partial \rho_i} = -\left[Y(\rho)\frac{\partial X}{\partial \rho_i}Y(\rho) + Y(\rho)\frac{\partial X_2}{\partial \rho_i}Y_2^T(\rho) + Y_2(\rho)\frac{\partial X_2^T}{\partial \rho_i}Y(\rho) + Y_2(\rho)\frac{\partial X_3}{\partial \rho_i}Y_2^T(\rho)\right]$$

for $i = 1, 2, \cdots, s$.

Proof: The proof for the first part of the lemma uses the matrix inversion lemma and Schur complements (see [Pac]). For the second part, define

$$W(\rho) = \begin{bmatrix} X(\rho) & X_2(\rho) \\ X_2^T(\rho) & X_3(\rho) \end{bmatrix}, \quad \text{and} \quad Z(\rho) = W^{-1}(\rho).$$

Differenting W^{-1} with respect to ρ_i gives

$$\frac{\partial Z}{\partial \rho_i} = -Z\frac{\partial W}{\partial \rho_i}Z.$$

The $(1,1)$ block of above equation is

$$\frac{\partial Y}{\partial \rho_i} = -\left[Y(\rho)\frac{\partial X}{\partial \rho_i}Y(\rho) + Y(\rho)\frac{\partial X_2}{\partial \rho_i}Y_2^T(\rho) + Y_2(\rho)\frac{\partial X_2^T}{\partial \rho_i}Y(\rho) + Y_2(\rho)\frac{\partial X_3}{\partial \rho_i}Y_2^T(\rho)\right],$$

which holds for $i = 1, 2, \cdots, s$. ∎

Theorem 4.3.1 *Given a compact set* $\mathcal{P} \subset \mathbf{R}^s$, *the performance level* $\gamma > 0$ *and the LPV system in (4.1.2) with restriction* $D_{11}(\rho) = 0$, *the Parameter-Dependent* γ*-Performance Problem is solvable if and only if there exist matrix functions* $X \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $Y \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$, *such that for all* $\rho \in \mathcal{P}$, $X(\rho) > 0$, $Y(\rho) > 0$, *and*

$$\begin{bmatrix} X(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)X(\rho) - \sum_{i=1}^{s} \pm\left(\nu_i\frac{\partial X}{\partial \rho_i}\right) - B_2(\rho)B_2^T(\rho) & X(\rho)C_{11}^T(\rho) & \gamma^{-1}B_1(\rho) \\ C_{11}(\rho)X(\rho) & -I_{n_{e1}} & 0 \\ \gamma^{-1}B_1^T(\rho) & 0 & -I_{n_d} \end{bmatrix} < 0, \tag{4.3.1.a}$$

$$\begin{bmatrix} \tilde{A}^T(\rho)Y(\rho) + Y(\rho)\tilde{A}(\rho) + \sum_{i=1}^{s} \pm\left(\nu_i\frac{\partial Y}{\partial \rho_i}\right) - C_2^T(\rho)C_2(\rho) & Y(\rho)B_{11}(\rho) & \gamma^{-1}C_1^T(\rho) \\ B_{11}^T(\rho)Y(\rho) & -I_{n_{d1}} & 0 \\ \gamma^{-1}C_1(\rho) & 0 & -I_{n_e} \end{bmatrix} < 0, \tag{4.3.1.b}$$

$$\begin{bmatrix} X(\rho) & \gamma^{-1}I_n \\ \gamma^{-1}I_n & Y(\rho) \end{bmatrix} \geq 0, \tag{4.3.1.c}$$

where $\hat{A}(\rho) := A(\rho) - B_2(\rho)C_{12}(\rho)$, $\tilde{A}(\rho) := A(\rho) - B_{12}(\rho)C_2(\rho)$.

If the conditions are satisfied, then by continuity and compactness, it is possible to perturb $X(\rho)$ *such that the two LMIs (4.3.1.a)–(4.3.1.b) still hold and* $Q(\rho) := Y(\rho) - \gamma^{-2}X^{-1}(\rho) > 0$ *uniformly on* $\mathcal{P}$. *Define*

$$\begin{aligned}
F(\rho) &:= -\left[B_2^T(\rho)X^{-1}(\rho) + D_{12}^T C_1(\rho)\right] \\
L(\rho) &:= -\left[Y^{-1}(\rho)C_2^T(\rho) + B_1(\rho)D_{21}^T\right] \\
H(\rho,\dot{\rho}) &:= -\left[X^{-1}(\rho)A_F(\rho) + A_F^T(\rho)X^{-1}(\rho) + \sum_{i=1}^{s}\left(\dot{\rho}_i \frac{\partial X^{-1}}{\partial \rho_i}\right) + C_F^T(\rho)C_F(\rho)\right. \\
&\qquad \left. + \gamma^{-2}X^{-1}(\rho)B_1(\rho)B_1^T(\rho)X^{-1}(\rho)\right],
\end{aligned}$$

with $A_F(\rho) := A(\rho) + B_2(\rho)F(\rho)$ *and* $C_F(\rho) := C_1(\rho) + D_{12}F(\rho)$. *Furthermore, let*

$$M(\rho,\dot{\rho}) := H(\rho,\dot{\rho}) + \gamma^2 Q(\rho)\left[-Q^{-1}(\rho)Y(\rho)L(\rho)D_{21} - B_1(\rho)\right]B_1^T(\rho)X^{-1}(\rho).$$

One n-dimensional, strictly proper controller $K_{\mathcal{P}}$ *with the state-space data in (4.1.3) that solves the feedback problem is given by*

$$\begin{aligned}
A_K(\rho,\dot{\rho}) &:= A(\rho) + B_2(\rho)F(\rho) + Q^{-1}(\rho)Y(\rho)L(\rho)C_2(\rho) - \gamma^{-2}Q^{-1}(\rho)M(\rho,\dot{\rho}) \\
B_K(\rho) &:= -Q^{-1}(\rho)Y(\rho)L(\rho) \\
C_K(\rho) &:= F(\rho) \\
D_K(\rho) &:= 0.
\end{aligned} \tag{4.3.2}$$

Proof: $\Rightarrow$ Let $W \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{(n+m)\times(n+m)})$ be the PDLF that satisfies the analysis test in Theorem 3.3.1 for the closed-loop system. Hence, W is bounded and uniformly positive definite over $\mathcal{P}$. Define $Z(\rho) := \gamma^{-2}W^{-1}(\rho)$. Clearly, Z is also continuously differentiable, bounded and uniformly positive definite over $\mathcal{P}$.

Partition W and Z as

$$W(\rho) = \begin{bmatrix} Y(\rho) & Y_2(\rho) \\ Y_2^T(\rho) & Y_3(\rho) \end{bmatrix}, \quad \text{and} \quad Z(\rho) = \begin{bmatrix} X(\rho) & X_2(\rho) \\ X_2^T(\rho) & X_3(\rho) \end{bmatrix},$$

where $X \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n}), Y \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$, $X_2 \in \mathcal{C}^1(\mathbf{R}^s, \mathbf{R}^{n\times m}), Y_2 \in \mathcal{C}^1(\mathbf{R}^s, \mathbf{R}^{n\times m})$, $X_3 \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{m\times m})$ and $Y_3 \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{m\times m})$. By Lemma 4.3.1, we get

$$\begin{bmatrix} X(\rho) & \gamma^{-1}I \\ \gamma^{-1}I & Y(\rho) \end{bmatrix} \geq 0$$

for all $\rho \in \mathcal{P}$.

To show necessity of inequalities (4.3.1.a) and (4.3.1.b), write the left hand side of equation (4.1.5) as

$$G(\rho,\beta) := R(\rho,\beta) + U(\rho)K(\rho,\beta)V^T(\rho) + V(\rho)K(\rho,\beta)U^T(\rho),$$

where

$$R := \begin{bmatrix} \begin{bmatrix} A^T & 0 \\ 0 & 0_m \end{bmatrix} W + W \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{s} \left(\beta_i \frac{\partial W}{\partial \rho_i}\right) & W \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} & \begin{matrix} \gamma^{-1} C_{11}^T & \gamma^{-1} C_{12}^T \\ 0 & 0 \end{matrix} \\ \begin{bmatrix} B_{11}^T & 0 \\ B_{12}^T & 0 \end{bmatrix} W & \begin{matrix} -I_{n_{d1}} & 0 \\ 0 & -I_{n_{d2}} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} \gamma^{-1} C_{11} & 0 \\ \gamma^{-1} C_{12} & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -I_{n_{e1}} & 0 \\ 0 & -I_{n_{e2}} \end{matrix} \end{bmatrix},$$

$$U := \begin{bmatrix} W \begin{bmatrix} 0 & B_2 \\ I_m & 0 \end{bmatrix} \\ 0 \quad 0 \\ 0 \quad 0 \\ 0 \quad 0 \\ 0 \quad \gamma^{-1} I_{n_{e2}} \end{bmatrix}, \quad V := \begin{bmatrix} 0 & C_2^T \\ I_m & 0 \\ 0 & 0 \\ 0 & I_{n_{d2}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}.$$

Define

$$U_\perp := \begin{bmatrix} \gamma X(\rho) & 0 & 0 & 0 \\ \gamma X_2^T(\rho) & 0 & 0 & 0 \\ 0 & I_{n_{d1}} & 0 & 0 \\ 0 & 0 & I_{n_{d2}} & 0 \\ 0 & 0 & 0 & I_{n_{e1}} \\ -B_2^T(\rho) & 0 & 0 & 0 \end{bmatrix}, \quad V_\perp := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{n_{d1}} & 0 & 0 \\ -C_2(\rho) & 0 & 0 & 0 \\ 0 & 0 & I_{n_{e1}} & 0 \\ 0 & 0 & 0 & I_{n_{e2}} \end{bmatrix}.$$

Note that for all $\rho \in \mathcal{P}$, $U_\perp^T U = 0$, $V_\perp^T V = 0$, and $[U \quad U_\perp]$, $[V \quad V_\perp]$ are full rank. Since both $U_\perp$ and $V_\perp$ are full column rank for all $\rho \in \mathcal{P}$, it is clear that if $G(\rho,\beta) < 0$ for all $\rho \in \mathcal{P}$ and $|\beta| \le \nu_i, \ i = 1, \cdots, s$, then

$$U_\perp^T(\rho) G(\rho,\beta) U_\perp(\rho) < 0 \qquad \text{and} \qquad V_\perp^T(\rho) G(\rho,\beta) V_\perp(\rho) < 0,$$

which implies that

$$U_\perp^T(\rho)R(\rho,\beta)U_\perp(\rho) < 0 \qquad \text{and} \qquad V_\perp^T(\rho)R(\rho,\beta)V_\perp(\rho) < 0$$

for all $\rho \in \mathcal{P}$ and $|\beta| \leq \nu_i$, $i = 1, \cdots, s$. Carrying out the algebraic manipulations, $U_\perp^T(\rho)R(\rho,\beta)U_\perp(\rho) < 0$ is equivalent to

$$\begin{bmatrix} \Omega(\rho,\beta) & \gamma^{-1}B_1(\rho) & X(\rho)C_{11}^T(\rho) \\ \gamma^{-1}B_1^T(\rho) & -I & 0 \\ C_{11}(\rho)X(\rho) & 0 & -I \end{bmatrix} < 0 \tag{4.3.3}$$

where

$$\begin{aligned} \Omega \;:=\; & \gamma^2\left(XY\hat{A}X + X\hat{A}^TYX + X\hat{A}^TY_2X_2^T + X_2Y_2^T\hat{A}X\right) \\ & \gamma^2\sum_{i=1}^{s}\beta_i\left[X\frac{\partial Y}{\partial\rho_i}X + X\frac{\partial Y_2}{\partial\rho_i}X_2^T + X_2\frac{\partial Y_2^T}{\partial\rho_i}X + X_2\frac{\partial Y_3}{\partial\rho_i}X_2^T\right] - B_2B_2^T. \end{aligned}$$

Using Lemma 4.3.1, we get

$$\gamma^2\left(X\frac{\partial Y}{\partial\rho_i}X + X\frac{\partial Y_2}{\partial\rho_i}X_2^T + X_2\frac{\partial Y_2^T}{\partial\rho_i}X + X_2\frac{\partial Y_3}{\partial\rho_i}X_2^T\right) = -\frac{\partial X}{\partial\rho_i}.$$

Furthermore $ZW = \gamma^{-2}I$, it follows that $YX + Y_2X_2^T = XY + X_2Y_2^T = \gamma^{-2}I$. This simplifies Ω to

$$\Omega(\rho,\beta) = \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}^T(\rho) - \sum_{i=1}^{s}\left(\beta_i\frac{\partial X}{\partial\rho_i}\right) - B_2(\rho)B_2^T(\rho).$$

Hence the condition in (4.3.3) is

$$\begin{bmatrix} \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}^T(\rho) - \sum_{i=1}^{s}\left(\beta_i\frac{\partial X}{\partial\rho_i}\right) - B_2(\rho)B_2^T(\rho) & \gamma^{-1}B_1(\rho) & X(\rho)C_{11}^T(\rho) \\ \gamma^{-1}B_1^T(\rho) & -I & 0 \\ C_{11}(\rho)X(\rho) & 0 & -I \end{bmatrix} < 0.$$

As $R(\rho,\beta)$ is an affine function of β, the above inequality is equivalent to

$$\begin{bmatrix} \hat{A}(\rho)X(\rho) + X(\rho)\hat{A}^T(\rho) - \sum_{i=1}^{s}\pm\left(\nu_i\frac{\partial X}{\partial\rho_i}\right) - B_2(\rho)B_2^T(\rho) & \gamma^{-1}B_1(\rho) & X(\rho)C_{11}^T(\rho) \\ \gamma^{-1}B_1^T(\rho) & -I & 0 \\ C_{11}(\rho)X(\rho) & 0 & -I \end{bmatrix} < 0,$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i,\ i = 1, 2, \cdots, s$. Simpler manipulations show that inequality $V_\perp(\rho) R(\rho, \beta) V_\perp^T(\rho) < 0$ is equivalent to

$$\begin{bmatrix} \tilde{A}^T(\rho) Y(\rho) + Y(\rho)\tilde{A}(\rho) + \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial Y}{\partial \rho_i}\right) - C_2^T(\rho) C_2(\rho) & Y(\rho) B_{11}(\rho) & \gamma^{-1}\tilde{C}^T(\rho) \\ B_{11}(\rho) Y(\rho) & -I & 0 \\ \gamma^{-1}\tilde{C}(\rho) & 0 & -I \end{bmatrix} < 0$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i,\ i = 1, 2, \cdots, s$.

$\Leftarrow$ For sufficiency, this direction uses the approach of [SamMN]. We verify the controller given in (4.3.2) satisfies the Parameter-Dependent γ-Performance Problem using the following function $W \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{2n \times 2n})$,

$$W(\rho) := \begin{bmatrix} Y(\rho) & -\left(Y(\rho) - \gamma^{-2} X^{-1}(\rho)\right) \\ -\left(Y(\rho) - \gamma^{-2} X^{-1}(\rho)\right) & Y(\rho) - \gamma^{-2} X^{-1}(\rho) \end{bmatrix}.$$

First, note that by Schur compliment, $W > 0$ for all $\rho \in \mathcal{P}$. Define

$$\begin{aligned} \Gamma(\rho, \beta) \quad &:= \quad A_{\text{clp}}^T(\rho, \beta) W(\rho) + W(\rho) A_{\text{clp}}(\rho, \beta) + \sum_{i=1}^{s} \left(\beta_i \frac{\partial W}{\partial \rho_i}\right) \\ &\quad + \gamma^{-2} C_{\text{clp}}^T(\rho) C_{\text{clp}}(\rho) + W(\rho) B_{\text{clp}}(\rho) B_{\text{clp}}^T(\rho) W(\rho) \end{aligned}$$

where the closed loop matrices $A_{\text{clp}}, B_{\text{clp}}$ and C_{clp} are defined in (4.1.4.). Partition Γ into $n \times n$ blocks $\Gamma_{11}, \Gamma_{12}, \Gamma_{22}$. Using the constant similarity transformation $T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ on Γ leads to

$$\begin{aligned} \tilde{\Gamma}(\rho, \beta) \quad &:= \quad T^T \Gamma(\rho, \beta) T \\ &= \quad \tilde{A}_{\text{clp}}^T(\rho, \beta) \tilde{W}(\rho) + \tilde{W}(\rho) \tilde{A}_{\text{clp}}(\rho, \beta) + \sum_{i=1}^{s} \left(\beta_i \frac{\partial \tilde{W}}{\partial \rho_i}\right) \\ &\quad + \gamma^{-2} \tilde{C}_{\text{clp}}^T(\rho) \tilde{C}_{\text{clp}}(\rho) + \tilde{W}(\rho) \tilde{B}_{\text{clp}}(\rho) \tilde{B}_{\text{clp}}^T(\rho) \tilde{W}(\rho) \end{aligned} \tag{4.3.4}$$

where

$$\begin{aligned} \tilde{A}_{\text{clp}}(\rho, \beta) \quad &:= \quad T^{-1} A_{\text{clp}}(\rho, \beta) T, \\ \tilde{B}_{\text{clp}}(\rho) \quad &:= \quad T^{-1} B_{\text{clp}}(\rho), \\ \tilde{C}_{\text{clp}}(\rho) \quad &:= \quad C_{\text{clp}}(\rho) T. \end{aligned}$$

and $\tilde{W}(\rho) := T^T W(\rho) T$. Partition $\tilde{\Gamma}$ into $n \times n$ blocks $\tilde{\Gamma}_{11}, \tilde{\Gamma}_{12}, \tilde{\Gamma}_{22}$. It is straightforward to verify that

$$\tilde{\Gamma}(\rho,\beta) = \begin{bmatrix} -\gamma^{-2}H(\rho,\beta) & -\gamma^{-2}H(\rho,\beta) \\ -\gamma^{-2}H(\rho,\beta) & \Gamma_{11}(\rho,\beta) - \gamma^{-2}H(\rho,\beta) \end{bmatrix}.$$

Note that

$$\begin{aligned} \tilde{\Gamma}_{11}(\rho,\beta) &= -\gamma^{-2}H(\rho,\beta) \\ &= \gamma^{-2}\Bigg[A_F^T(\rho)X^{-1}(\rho) + X^{-1}(\rho)A_F(\rho) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial X^{-1}}{\partial \rho_i}\right) \\ &\qquad + C_F^T(\rho)C_F(\rho) + \gamma^{-2}X^{-1}(\rho)B_1(\rho)B_1^T(\rho)X^{-1}(\rho)\Bigg] \end{aligned}$$

with $A_F(\rho) = A(\rho) + B_2(\rho)F(\rho), C_F^T(\rho) = \left[C_{11}^T(\rho) \;\; C_{12}^T(\rho) + F^T(\rho)\right]$. So $\tilde{\Gamma}_{11}(\rho,\beta)$ is negative definite for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i, \; i = 1, 2, \cdots, s$ by Theorem 4.2.1. Furthermore

$$\begin{aligned} &\tilde{\Gamma}_{22}(\rho,\beta) - \tilde{\Gamma}_{12}^T(\rho,\beta)\tilde{\Gamma}_{11}^{-1}(\rho,\beta)\tilde{\Gamma}_{12}(\rho,\beta) \\ &= \Gamma_{11}(\rho,\beta) - \gamma^{-2}H(\rho,\beta) - \left[-\gamma^{-2}H(\rho,\beta)\right]\left[-\gamma^{-2}H(\rho,\beta)\right]^{-1}\left[-\gamma^{-2}H(\rho,\beta)\right] \\ &= \Gamma_{11}(\rho,\beta) \\ &= Y(\rho)A_L(\rho) + A_L^T(\rho)Y(\rho) + \sum_{i=1}^{s}\left(\beta_i\frac{\partial Y}{\partial \rho_i}\right) + \gamma^{-2}C_1^T(\rho)C_1(\rho) + Y(\rho)B_L(\rho)B_L^T(\rho)Y(\rho) \end{aligned}$$

with $A_L(\rho) = A(\rho) + L(\rho)C_2(\rho), B_L(\rho) = [B_{11}(\rho) \;\; B_{12}(\rho) + L(\rho)]$. This quantity is negative definite for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i, \; i = 1, 2, \cdots, s$ by the dual of Theorem 4.2.1. Combine $\tilde{\Gamma}_{11} < 0$ and $\tilde{\Gamma}_{22} - \tilde{\Gamma}_{12}^T\tilde{\Gamma}_{11}^{-1}\tilde{\Gamma}_{12} < 0$, we get $\tilde{\Gamma} < 0$ by Schur complement arguments. From equation (4.3.4), $\Gamma(\rho,\beta) < 0$ for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i, \; i = 1, 2, \cdots, s$ as desired. ∎

4.3.2 Non-zero $D_{11}(\rho)$ case

In order to prove the general case of $D_{11}(\rho) \neq 0$, we need the following lemma (adopted from [DavKW] & [Doy]) to convert $D_{11}(\rho)$ term to required form.

Lemma 4.3.2 *Given matrices A, B and C with compatible dimensions, then*

$$\min_X \bar{\sigma}\begin{bmatrix} A & B \\ C & X \end{bmatrix} = \max\left\{\bar{\sigma}\,[A \;\; B]\,, \bar{\sigma}\begin{bmatrix} A \\ C \end{bmatrix}\right\} =: \gamma_0.$$

For any $\gamma > \gamma_0$, *one of the solution for the inequality*

$$\bar{\sigma}\begin{bmatrix} A & B \\ C & X \end{bmatrix} \le \gamma$$

is given by $X = -C\left(\gamma^2 I - A^T A\right)^{-1} A^T B$.

Proof: see [DavKW] & [Doy]. ∎

For notational purposes, denote

$$\begin{bmatrix} D_{111\cdot}(\rho) \\ D_{112\cdot}(\rho) \end{bmatrix} := \left[\begin{array}{cc} D_{1111}(\rho) & D_{1112}(\rho) \\ \hline D_{1121}(\rho) & D_{1122}(\rho) \end{array}\right],$$

$$[D_{11\cdot 1}(\rho) \quad D_{11\cdot 2}(\rho)] := \left[\begin{array}{c|c} D_{1111}(\rho) & D_{1112}(\rho) \\ D_{1121}(\rho) & D_{1122}(\rho) \end{array}\right].$$

Now we can give the complete solvability condition for the Parameter-Dependent γ-Performance Problem with $D_{11}(\rho) \neq 0$.

Theorem 4.3.2 *Given a compact set* $\mathcal{P}$, *the performance level* $\gamma > 0$ *and the LPV system in (4.1.2), the Parameter-Dependent* γ*-Performance Problem is solvable if and only if there exist matrix functions* $X \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $Y \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$, *such that for all* $\rho \in \mathcal{P}$, $X(\rho), Y(\rho) > 0$, *and*

$$\begin{bmatrix} X(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)X(\rho) - \sum_{i=1}^{s} \pm\left(\nu_i \frac{\partial X}{\partial \rho_i}\right) - B_2(\rho)B_2^T(\rho) & X(\rho)C_{11}^T(\rho) & \gamma^{-1}\hat{B}(\rho) \\ C_{11}(\rho)X(\rho) & -I_{n_{e1}} & \gamma^{-1}D_{111\cdot}(\rho) \\ \gamma^{-1}\hat{B}^T(\rho) & \gamma^{-1}D_{111\cdot}^T(\rho) & -I_{n_d} \end{bmatrix} < 0, \tag{4.3.5.a}$$

$$\begin{bmatrix} \tilde{A}^T(\rho)Y(\rho) + Y(\rho)\tilde{A}(\rho) + \sum_{i=1}^{s} \pm\left(\nu_i \frac{\partial Y}{\partial \rho_i}\right) - C_2^T(\rho)C_2(\rho) & Y(\rho)B_{11}(\rho) & \gamma^{-1}\tilde{C}^T(\rho) \\ B_{11}^T(\rho)Y(\rho) & -I_{n_{d1}} & \gamma^{-1}D_{11\cdot 1}^T(\rho) \\ \gamma^{-1}\tilde{C}(\rho) & \gamma^{-1}D_{11\cdot 1}(\rho) & -I_{n_e} \end{bmatrix} < 0, \tag{4.3.5.b}$$

$$\begin{bmatrix} X(\rho) & \gamma^{-1}I_n \\ \gamma^{-1}I_n & Y(\rho) \end{bmatrix} \ge 0, \tag{4.3.5.c}$$

where

$$\begin{aligned}
\hat{A}(\rho) &:= A(\rho) - B_2(\rho)C_{12}(\rho), \qquad \hat{B}(\rho) := B_1(\rho) - B_2(\rho)D_{112\cdot}(\rho),\\
\tilde{A}(\rho) &:= A(\rho) - B_{12}(\rho)C_2(\rho), \qquad \tilde{C}(\rho) := C_1(\rho) - D_{11\cdot 2}(\rho)C_2(\rho).
\end{aligned}$$

If the conditions are satisfied, then by continuity and compactness, it is possible to perturb $X(\rho)$ such that the two LMIs (4.3.5.a)-(4.3.5.b) still hold and $Q(\rho) := Y(\rho) - \gamma^{-2}X^{-1}(\rho) > 0$ uniformly on $\mathcal{P}$. Define

$$\begin{aligned}
\Omega(\rho) &:= -D_{1122}(\rho) - D_{1121}(\rho)\left[\gamma^2 I_{n_{d1}} - D_{1111}^T(\rho)D_{1111}(\rho)\right]^{-1} D_{1111}^T(\rho)D_{1112}(\rho),\\
\bar{A}(\rho) &:= A(\rho) + B_2(\rho)\Omega(\rho)C_2(\rho),\\
\bar{B}_1(\rho) &:= B_1(\rho) + B_2(\rho)\Omega(\rho)D_{21},\\
\bar{C}_1(\rho) &:= C_1(\rho) + D_{12}\Omega(\rho)C_2(\rho),\\
\bar{D}_{11}(\rho) &:= D_{11}(\rho) + D_{12}\Omega(\rho)D_{21},\\
D_h(\rho) &:= \left[I_{n_e} - \gamma^{-2}\bar{D}_{11}(\rho)\bar{D}_{11}^T(\rho)\right]^{-1},\\
D_t(\rho) &:= \left[I_{n_d} - \gamma^{-2}\bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)\right]^{-1},
\end{aligned}$$

and

$$\begin{aligned}
F(\rho) &:= -\left(D_{12}^T D_h(\rho)D_{12}\right)^{-1}\\
&\quad \star\left[\left(B_2(\rho) + \gamma^{-2}\bar{B}_1(\rho)\bar{D}_{11}^T(\rho)D_h(\rho)D_{12}\right)^T X^{-1}(\rho) + D_{12}^T D_h(\rho)\bar{C}_1(\rho)\right],\\
L(\rho) &:= -\left[Y^{-1}(\rho)\left(C_2(\rho) + \gamma^{-2}D_{21}D_t(\rho)\bar{D}_{11}^T(\rho)\bar{C}_1(\rho)\right)^T + \bar{B}_1(\rho)D_t(\rho)D_{21}^T\right]\\
&\quad \star\left(D_{21}D_t(\rho)D_{21}^T\right)^{-1},\\
H(\rho,\dot{\rho}) &:= -\left[X^{-1}(\rho)A_F(\rho) + A_F^T(\rho)X^{-1}(\rho) + \sum_{i=1}^{s}\left(\dot{\rho}_i\frac{\partial X^{-1}}{\partial\rho_i}\right) + C_F^T(\rho)C_F(\rho)\right.\\
&\quad + \left(X^{-1}(\rho)\bar{B}_1(\rho) + C_F^T(\rho)\bar{D}_{11}(\rho)\right)\\
&\quad \left.\star\left(\gamma^2 I - \bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)\right)^{-1}\left(\bar{B}_1^T(\rho)X^{-1}(\rho) + \bar{D}_{11}^T(\rho)C_F(\rho)\right)\right],
\end{aligned}$$

with $A_F(\rho) := \bar{A}(\rho) + B_2(\rho)F(\rho)$ and $C_F(\rho) := \bar{C}_1(\rho) + D_{12}(\rho)F(\rho)$. Furthermore, let

$$\begin{aligned}
M(\rho,\dot{\rho}) &:= H(\rho,\dot{\rho}) + F^T(\rho)\left[B_2^T(\rho)X^{-1}(\rho) + D_{12}^T\left(\bar{C}_1(\rho) + D_{12}F(\rho)\right)\right]\\
&\quad + \left[\gamma^2 Q(\rho)\left(-Q^{-1}(\rho)Y(\rho)L(\rho)D_{21} - \bar{B}_1(\rho)\right) + F^T(\rho)D_{12}^T\bar{D}_{11}(\rho)\right]\\
&\quad \star\left[\gamma^2 I - \bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)\right]^{-1}\left[\bar{B}_1^T(\rho)X^{-1}(\rho) + \bar{D}_{11}^T(\rho)\left(\bar{C}_1(\rho) + D_{12}(\rho)F(\rho)\right)\right].
\end{aligned}$$

One n-dimensional, proper controller $K_{\mathcal{P}}$ with the state-space data in (4.1.3) that solves the feedback problem is given by

$$\begin{aligned}
A_K(\rho,\dot{\rho}) &:= \bar{A}(\rho) + B_2(\rho)F(\rho) + Q^{-1}(\rho)Y(\rho)L(\rho)C_2(\rho) - \gamma^{-2}Q^{-1}(\rho)M(\rho,\dot{\rho}),\\
B_K(\rho) &:= -Q^{-1}(\rho)Y(\rho)L(\rho),\\
C_K(\rho) &:= F(\rho),\\
D_K(\rho) &:= \Omega(\rho).
\end{aligned} \tag{4.3.6}$$

Proof: The idea behind the proof is to transform the state space data to the case $D_{11} = 0$, then employ Theorem 4.3.1 to get conclusion.

Using the variable transformation $v = u - \Omega(\rho)y$, the system becomes

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} &= \begin{bmatrix} A(\rho) + B_2(\rho)\Omega(\rho)C_2(\rho) & B_1(\rho) + B_2(\rho)\Omega(\rho)D_{21} & B_2(\rho) \\ C_1(\rho) + D_{12}\Omega(\rho)C_2(\rho) & D_{11}(\rho) + D_{12}\Omega(\rho)D_{21} & D_{12} \\ C_2(\rho) & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ v \end{bmatrix} \\
&:= \begin{bmatrix} \bar{A}(\rho) & \bar{B}_1(\rho) & \bar{B}_2(\rho) \\ \bar{C}_1(\rho) & \bar{D}_{11}(\rho) & \bar{D}_{12} \\ \bar{C}_2(\rho) & \bar{D}_{21} & \bar{D}_{22} \end{bmatrix} \begin{bmatrix} x \\ d \\ v \end{bmatrix}.
\end{aligned} \tag{4.3.7}$$

From Lemma 4.3.2, we know that the LPV system in (4.1.2) has induced $\mathbf{L}_2$-norm less than γ only if $\bar{\sigma}\left[\bar{D}_{11}(\rho)\right] < \gamma$. So for Parameter-Dependent γ-Performance Problem, $\left[I - \gamma^{-2}\bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)\right]$ and $\left[I - \gamma^{-2}\bar{D}_{11}(\rho)\bar{D}_{11}^T(\rho)\right]$ are both invertible. Define the following unity, parameter-dependent transformation [SafLC]

$$\begin{bmatrix} \gamma^{-1}\grave{e} \\ d \end{bmatrix} = \begin{bmatrix} -\gamma^{-1}\bar{D}_{11}(\rho) & \left[I - \gamma^{-2}\bar{D}_{11}(\rho)\bar{D}_{11}^T(\rho)\right]^{\frac{1}{2}} \\ \left[I - \gamma^{-2}\bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)\right]^{\frac{1}{2}} & \gamma^{-1}\bar{D}_{11}^T(\rho) \end{bmatrix} \begin{bmatrix} \grave{d} \\ \gamma^{-1}e \end{bmatrix} := \mathcal{T}(\rho) \begin{bmatrix} \grave{d} \\ \gamma^{-1}e \end{bmatrix}.$$

Re-arranging the inputs and outputs of system equation (4.3.7), and scale the outputs by γ^{-1}, we get

$$\begin{bmatrix} \gamma^{-1}e \\ \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} \gamma^{-1}\bar{D}_{11}(\rho) & \gamma^{-1}\bar{C}_1(\rho) & \gamma^{-1}D_{12} \\ \bar{B}_1(\rho) & \bar{A}(\rho) & B_2(\rho) \\ D_{21} & C_2(\rho) & 0 \end{bmatrix} \begin{bmatrix} d \\ x \\ v \end{bmatrix} := \mathcal{B}(\rho) \begin{bmatrix} d \\ x \\ v \end{bmatrix}.$$

Applying $\mathcal{T}(\rho)$ to the top of $\mathcal{B}(\rho)$, $\mathcal{F}_\ell(\mathcal{T}(\rho),\mathcal{B}(\rho))$ is given by

$$\begin{bmatrix} \mathcal{F}_\ell(\mathcal{T}(\rho),\mathcal{B}_{11}(\rho)) & \mathcal{T}_{12}(\rho)(I-\mathcal{B}_{11}(\rho)\mathcal{T}_{22}(\rho))^{-1}\mathcal{B}_{12}(\rho) \\ \mathcal{B}_{21}(\rho)(I-\mathcal{T}_{22}(\rho)\mathcal{B}_{11}(\rho))^{-1}\mathcal{T}_{21}(\rho) & \mathcal{F}_u(\mathcal{B}(\rho),\mathcal{T}_{22}(\rho)) \end{bmatrix},$$

where

$$\begin{aligned}
\mathcal{F}_\ell(\mathcal{T},\mathcal{B}_{11}) &= 0 := \gamma^{-1}\grave{D}_{11}, \\
\mathcal{T}_{12}\left(I-\mathcal{B}_{11}\mathcal{T}_{22}\right)^{-1}\mathcal{B}_{12} &= \gamma^{-1}\left[\left(I-\gamma^{-2}\bar{D}_{11}\bar{D}_{11}^T\right)^{-\frac{1}{2}}\bar{C}_1 \quad \left(I-\gamma^{-2}\bar{D}_{11}\bar{D}_{11}^T\right)^{-\frac{1}{2}}D_{12}\right] \\
&:= \left[\gamma^{-1}\grave{C}_1 \quad \gamma^{-1}\grave{D}_{12}\right], \\
\mathcal{B}_{21}(I-\mathcal{T}_{22}\mathcal{B}_{11})^{-1}\mathcal{T}_{21} &= \begin{bmatrix} \bar{B}_1\left(I-\gamma^{-2}\bar{D}_{11}^T\bar{D}_{11}\right)^{-\frac{1}{2}} \\ D_{21}\left(I-\gamma^{-2}\bar{D}_{11}^T\bar{D}_{11}\right)^{-\frac{1}{2}} \end{bmatrix} \\
&:= \begin{bmatrix} \grave{B}_1 \\ \grave{D}_{21} \end{bmatrix}, \\
\mathcal{F}_u(\mathcal{B},\mathcal{T}_{22}) &= \begin{bmatrix} \bar{A}+\gamma^{-2}\bar{B}_1\bar{D}_{11}^T D_h\bar{C}_1 & B_2+\gamma^{-2}\bar{B}_1\bar{D}_{11}^T D_h D_{12} \\ C_2+\gamma^{-2}D_{21}\bar{D}_{11}^T D_h\bar{C}_1 & 0 \end{bmatrix} \\
&:= \begin{bmatrix} \grave{A} & \grave{B}_2 \\ \grave{C}_2 & \grave{D}_{22} \end{bmatrix}.
\end{aligned}$$

So the transformed system is given by

$$\begin{bmatrix} \dot{x} \\ \grave{e} \\ y \end{bmatrix} = \begin{bmatrix} \grave{A}(\rho) & \grave{B}_1(\rho) & \grave{B}_2(\rho) \\ \grave{C}_1(\rho) & \grave{D}_{11}(\rho) & \grave{D}_{12}(\rho) \\ \grave{C}_2(\rho) & \grave{D}_{21}(\rho) & \grave{D}_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ \grave{d} \\ v \end{bmatrix}. \tag{4.3.8}$$

By unitary property of $\mathcal{T}$, we get

$$\gamma^{-2}\|e\|^2+\|\grave{d}\|^2=\gamma^{-2}\|\grave{e}\|^2+\|d\|^2, \tag{4.3.9}$$

But now $\grave{D}_{12},\grave{D}_{21}$ are not in the standard $[0 \quad I]^T$ and $[0 \quad I]$ form any more. So we need to rescale the inputs and outputs of the system (4.3.8). Define

$$Q_{12} := \begin{bmatrix} \left(D_{12\perp}^T D_h^{-1} D_{12\perp}\right)^{-\frac{1}{2}} D_{12\perp}^T D_h^{-\frac{1}{2}} \\ \left(D_{12}^T D_h D_{12}\right)^{-\frac{1}{2}} D_{12}^T D_h^{\frac{1}{2}} \end{bmatrix},$$

$$R_{12} := \left(D_{12}^T D_h D_{12}\right)^{-\frac{1}{2}},$$
$$Q_{21} := \left[D_t^{-\frac{1}{2}} D_{21\perp}^T \left(D_{21\perp} D_t^{-1} D_{21\perp}^T\right)^{-\frac{1}{2}} \quad D_t^{\frac{1}{2}} D_{21}^T \left(D_{21} D_t D_{21}^T\right)^{-\frac{1}{2}}\right],$$
$$R_{21} := \left(D_{21} D_t D_{21}^T\right)^{-\frac{1}{2}},$$

where the matrices $D_{12\perp}$ and $D_{21\perp}$ are such that $D_{12\perp}^T D_{12} = 0$, $D_{21\perp} D_{21}^T = 0$, and $[D_{21\perp} \;\; D_{12}]$, $\left[D_{21\perp}^T \;\; D_{21}^T\right]$ are full rank. Scale the inputs $\grave{d}$ and v by Q_{21} and R_{12}, and the outputs $\grave{e}$ and y by Q_{12} and R_{21}, it is easy to show that

$$Q_{12}\grave{D}_{12}R_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad R_{21}\grave{D}_{21}Q_{21} = [0 \;\; I].$$

Then we have the new state-space data as

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \check{e} \\ \check{y} \end{bmatrix} &= \begin{bmatrix} \grave{A}(\rho) & \grave{B}_1(\rho)Q_{21}(\rho) & \grave{B}_2(\rho)R_{12}(\rho) \\ Q_{12}(\rho)\grave{C}_1(\rho) & 0 & D_{12} \\ R_{21}(\rho)\grave{C}_2(\rho) & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ \check{d} \\ \check{v} \end{bmatrix} \\
&:= \begin{bmatrix} \check{A}(\rho) & \check{B}_1(\rho) & \check{B}_2(\rho) \\ \check{C}_1(\rho) & \check{D}_{11}(\rho) & \check{D}_{12}(\rho) \\ \check{C}_2(\rho) & \check{D}_{21}(\rho) & \check{D}_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ \check{d} \\ \check{v} \end{bmatrix}.
\end{aligned} \tag{4.3.10}$$

Note that Q_{12}, Q_{21} are unitary matrices, so

$$\|\check{e}\|^2 = \|\grave{e}\|^2, \quad \|\check{d}\|^2 = \|\grave{d}\|^2. \tag{4.3.11}$$

Now equation (4.3.10) is in the standard form with Assumptions $(B1) - (B3)$ satisfied. From equations (4.3.9) and (4.3.11), $\|e\| \le \gamma\|d\|$ if and only if $\|\check{e}\| \le \gamma\|\check{d}\|$. Using Theorem 4.3.1, we get the necessary and sufficient conditions of Parameter-Dependent γ-Performance Problem for modified LPV system in (4.3.10) as follows:

$$\begin{bmatrix} X\left(\check{A} - \check{B}_2\check{C}_{12}\right)^T + \left(\check{A} - \check{B}_2\check{C}_{12}\right)X - \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial X}{\partial \rho_i}\right) - \check{B}_2\check{B}_2^T & X\check{C}_{11}^T & \gamma^{-1}\check{B}_1 \\ \check{C}_{11}X & -I_{n_{e1}} & 0 \\ \gamma^{-1}\check{B}_1^T & 0 & -I_{n_d} \end{bmatrix} < 0 \tag{4.3.12.a}$$

$$\begin{bmatrix} Y\left(\check{A}-\check{B}_{12}\check{C}_2\right)+\left(\check{A}-\check{B}_{12}\check{C}_2\right)^T Y+\sum_{i=1}^{s}\pm\left(\nu_i\dfrac{\partial Y}{\partial \rho_i}\right)-\check{C}_2^T\check{C}_2 & Y\check{B}_{11} & \gamma^{-1}\check{C}_1^T \\ \check{B}_{11}^T Y & -I_{n_{d1}} & 0 \\ \gamma^{-1}\check{C}_1 & 0 & -I_{n_e} \end{bmatrix} < 0 \qquad (4.3.12.b)$$

$$\begin{bmatrix} X & \gamma^{-1}I_n \\ \gamma^{-1}I_n & Y \end{bmatrix} \geq 0 \qquad (4.3.12.c)$$

After tedious manipulation, we can show that equations (4.3.12.) are equivalent to equations (4.3.5.).

Furthermore, the controller formula for the transformed system (4.3.10) is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_k \\ \check{v} \end{bmatrix} &= \begin{bmatrix} \check{A}_k & \check{B}_k \\ \check{C}_k & \check{D}_k \end{bmatrix} \begin{bmatrix} x_k \\ \check{y} \end{bmatrix} \\ &= \begin{bmatrix} \check{A}+\check{B}_2\check{F}+Q^{-1}Y\check{L}\check{C}_2-\gamma^{-2}Q^{-1}\check{M} & -Q^{-1}Y\check{L} \\ \check{F} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \check{y} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \check{F}(\rho) &= -\left[\check{B}_2^T(\rho)X^{-1}(\rho)+D_{12}^T\check{C}_1(\rho)\right], \\ \check{L}(\rho) &= -\left[Y^{-1}(\rho)\check{C}_2^T(\rho)+\check{B}_1(\rho)D_{21}^T\right], \\ \check{H}(\rho,\dot{\rho}) &= -\left[X^{-1}(\rho)\check{A}_F(\rho)+\check{A}_F^T(\rho)X^{-1}(\rho)+\sum_{i=1}^{s}\left(\dot{\rho}_i\frac{\partial X^{-1}}{\partial \rho_i}\right)+\check{C}_F^T(\rho)\check{C}_F(\rho)\right. \\ &\quad \left. +X^{-1}(\rho)\check{B}_1(\rho)\check{B}_1^T(\rho)X^{-1}(\rho)\right], \end{aligned}$$

with $\check{A}_F(\rho)=\check{A}(\rho)+\check{B}_2(\rho)\check{F}(\rho)$ and $\check{C}_F(\rho)=\check{C}_1(\rho)+D_{12}\check{F}(\rho)$, and

$$\check{M}(\rho,\dot{\rho})=\check{H}(\rho,\dot{\rho})+\gamma^2Q(\rho)\left(-Q^{-1}(\rho)Y(\rho)\check{L}(\rho)D_{21}-\check{B}_1(\rho)\right)\check{B}_1^T(\rho)X^{-1}(\rho).$$

It is easy to check that

$$\begin{aligned} R_{12}(\rho)\check{F}(\rho) &= -R_{12}(\rho)\left[\check{B}_2^T(\rho)X^{-1}(\rho)+\check{C}_{12}(\rho)\right] \\ &= -\left(D_{12}^TD_h(\rho)D_{12}\right)^{-1} \\ &\quad \star\left[\left(B_2(\rho)+\gamma^{-2}\bar{B}_1(\rho)\bar{D}_{11}^T(\rho)D_h(\rho)D_{12}\right)^T X^{-1}(\rho)+D_{12}^TD_h(\rho)\bar{C}_1(\rho)\right] \end{aligned}$$

$$
\begin{aligned}
&= F(\rho), \\
\check{L}(\rho)R_{21}(\rho) &= -\left[Y^{-1}(\rho)\check{C}_2^T(\rho) + \check{B}_{12}(\rho)\right] R_{21}(\rho) \\
&= -\left[Y^{-1}(\rho)\left(C_2(\rho) + \gamma^{-2}D_{21}D_t(\rho)\bar{D}_{11}^T(\rho)\bar{C}_1(\rho)\right)^T + \bar{B}_1(\rho)D_t(\rho)D_{21}\right] \\
&\quad \star \left(D_{21}D_t(\rho)D_{21}^T\right)^{-1} \\
&= L(\rho),
\end{aligned}
$$

and

$$
\begin{aligned}
\check{H}(\rho,\dot{\rho}) &= -\left[X^{-1}(\rho)A_F(\rho) + A_F^T(\rho)X^{-1}(\rho) + \sum_{i=1}^{s}\left(\dot{\rho}_i\frac{\partial X^{-1}}{\partial \rho_i}\right) + C_F^T(\rho)C_F(\rho)\right. \\
&\quad + \left(X^{-1}(\rho)\bar{B}_1(\rho) + C_F^T(\rho)\bar{D}_{11}(\rho)\right) \\
&\quad \left.\star \left(\gamma^2 I - \bar{D}_{11}^T(\rho)\bar{D}_{11}(\rho)\right)^{-1}\left(\bar{B}_1^T(\rho)X^{-1}(\rho) + \bar{D}_{11}^T(\rho)C_F(\rho)\right)\right] \\
&= H(\rho,\dot{\rho}),
\end{aligned}
$$

$$
\begin{aligned}
&\check{M}(\rho,\dot{\rho}) - \gamma^2 Q(\rho)\left[\gamma^{-2}\bar{B}_1(\rho)\bar{D}_{11}^T(\rho)D_h(\rho)\left(\bar{C}_1(\rho) + D_{12}F(\rho)\right)\right. \\
&\qquad \left. + \gamma^{-2}Q^{-1}(\rho)Y(\rho)L(\rho)D_{21}\bar{D}_{11}^T(\rho)D_h(\rho)\bar{C}_1(\rho)\right] \\
&= M(\rho,\dot{\rho}).
\end{aligned}
$$

So the controller for the original LPV system (4.1.2) is given by

$$
\begin{bmatrix} \dot{x} \\ u \end{bmatrix} = \begin{bmatrix} \bar{A} + B_2F + Q^{-1}YLC_2 - \gamma^{-2}Q^{-1}M & -Q^{-1}YL \\ F & \Omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

as desired. ∎

Theorem 4.3.2 formulates the necessary and sufficient conditions for the solvability of the Parameter-Dependent γ-Performance Problem, and gives one parametrically-dependent stabilizing output-feedback controller. All of the inequalities in (4.3.5.) are in the form of LMIs of continuously differentiable matrix functions $X(\rho)$, $Y(\rho)$, which lead to infinite dimensional convex feasibility problem. These conditions are parallel to the results in [DoyGKF], [GloD], [Gah1]. Specifically, equation (4.3.5.a) is for state-feedback, (4.3.5.b) for output estimation and (4.3.5.c) as coupling condition.

4.4 Computational Considerations

In this section, we discuss computational issues of the solution to the Parameter-Dependent γ-Performance Problem given by Theorem 4.3.2. First, we use an "ad-hoc" approach to convert the infinite convex feasibility conditions in (4.3.5.) to finite dimensional LMIs, which retain the convex property of original conditions. Then we discuss sufficient gradding density which guarantees global solving of the resulted LMIs over the whole parameter set.

4.4.1 Convex Computational Algorithm

Several approaches to solve the LMIs in Theorem 4.3.2 can be proposed. One way is to parameterize infinite-dimensional function space using finite number of basis functions. Such a parameterization leads to a sufficient condition for the Parameter-Dependent γ-Performance Problem.

Theorem 4.4.1 *Given finite number of scalar, continuously differentiable functions* $\{f_i\}_{i=1}^N$ *and* $\{g_i\}_{i=1}^N$ *with the parameterization*

$$X(\rho) := \sum_{i=1}^N f_i(\rho) X_i, \qquad Y(\rho) := \sum_{i=1}^N g_i(\rho) Y_i. \tag{4.4.1}$$

The Parameter-Dependent γ*-Performance Problem is solvable if there exist matrices* $\{X_i\}_{i=1}^N$, $X_i \in \mathcal{S}^{n\times n}$ *and* $\{Y_i\}_{i=1}^N$, $Y_i \in \mathcal{S}^{n\times n}$, *such that for all* $\rho \in \mathcal{P}$, $X(\rho) > 0$, $Y(\rho) > 0$ *and*

$$\begin{bmatrix} \begin{array}{c} \sum_{i=1}^N f_i(\rho)\left(X_i \hat{A}^T(\rho) + \hat{A}(\rho) X_i\right) \\ -\sum_{j=1}^s \pm\left(\nu_j \sum_{i=1}^N \frac{\partial f_i}{\partial \rho_j} X_i\right) - B_2(\rho) B_2^T(\rho) \end{array} & \sum_{i=1}^N f_i(\rho) X_i C_{11}^T(\rho) & \gamma^{-1}\hat{B}(\rho) \\ C_{11}(\rho)\sum_{i=1}^N f_i(\rho) X_i & -I_{n_{e1}} & \gamma^{-1} D_{111\cdot}(\rho) \\ \gamma^{-1}\hat{B}^T(\rho) & \gamma^{-1} D_{111\cdot}^T(\rho) & -I_{n_d} \end{bmatrix} < 0, \tag{4.4.2.a}$$

$$\begin{bmatrix} \begin{array}{c} \sum_{i=1}^N g_i(\rho)\left(\tilde{A}^T(\rho) Y_i + Y_i \tilde{A}(\rho)\right) \\ +\sum_{j=1}^s \pm\left(\nu_j \sum_{i=1}^N \frac{\partial g_i}{\partial \rho_j} Y_i\right) - C_2^T(\rho) C_2(\rho) \end{array} & \sum_{i=1}^N g_i(\rho) Y_i B_{11}(\rho) & \gamma^{-1}\tilde{C}^T(\rho) \\ B_{11}^T(\rho)\sum_{i=1}^N g_i(\rho) Y_i & -I_{n_{d1}} & \gamma^{-1} D_{11\cdot 1}^T(\rho) \\ \gamma^{-1}\tilde{C}(\rho) & \gamma^{-1} D_{11\cdot 1}(\rho) & -I_{n_e} \end{bmatrix} < 0, \tag{4.4.2.b}$$

$$\begin{bmatrix} \sum_{i=1}^{N} f_i(\rho) X_i & \gamma^{-1} I \\ \gamma^{-1} I & \sum_{i=1}^{N} g_i(\rho) Y_i \end{bmatrix} < 0. \tag{4.4.2.c}$$

where

$$\begin{aligned} \hat{A}(\rho) &:= A(\rho) - B_2(\rho)C_{12}(\rho), \quad \hat{B}(\rho) := B_1(\rho) - B_2(\rho)D_{112\cdot}(\rho), \\ \tilde{A}(\rho) &:= A(\rho) - B_{12}(\rho)C_2(\rho), \quad \tilde{C}(\rho) := C_1(\rho) - D_{11\cdot 2}(\rho)C_2(\rho). \end{aligned}$$

Using such $X(\rho)$ and $Y(\rho)$, one admissible controller is formulated by (4.3.6).

Proof: Because the parameterization given in (4.4.1) is a candidate for continuously differentiable functions in Theorem 4.3.2, the claim is clearly true. ∎

By parameterizing functions $X(\rho), Y(\rho)$ with some pre-selected functions $\{f_i\}_{i=1}^N$, $\{g_i\}_{i=1}^N$, we come up with conditions (4.4.2.), which consists of $2^{s+1} + 1$ LMIs of matrix variables $X_1, Y_1, \cdots, X_N, Y_N$. But such parameterization restricts the functions we search over, and it may lead to some conservatism because the solvability condition (4.3.5.) now depends on the basis functions you pick.

Next we will show the convexity of conditions in (4.4.2.). Define following matrix functions

$$Ric_X^{\infty}(X_1, \cdots, X_N, \rho) :=$$

$$\begin{bmatrix} \begin{array}{l} \sum_{i=1}^{N} f_i(\rho)\left(X_i \hat{A}^T(\rho) + \hat{A}(\rho) X_i\right) \\ -\sum_{j=1}^{s} \pm \left(\nu_j \sum_{i=1}^{N} \frac{\partial f_i}{\partial \rho_j} X_i\right) - B_2(\rho) B_2^T(\rho) \end{array} & \sum_{i=1}^{N} f_i(\rho) X_i C_{11}^T(\rho) & \gamma^{-1}\hat{B}(\rho) \\ C_{11}(\rho) \sum_{i=1}^{N} f_i(\rho) X_i & -I_{n_{e1}} & \gamma^{-1} D_{111\cdot}(\rho) \\ \gamma^{-1}\hat{B}^T(\rho) & \gamma^{-1} D_{111\cdot}^T(\rho) & -I_{n_d} \end{bmatrix},$$

$$Ric_Y^{\infty}(Y_1, \cdots, Y_N, \rho) :=$$

$$\begin{bmatrix} \sum_{i=1}^{N} g_i(\rho)\left(\tilde{A}^T(\rho)Y_i + Y_i\tilde{A}(\rho)\right) + \sum_{j=1}^{s} \pm \left(\nu_j \sum_{i=1}^{N} \frac{\partial g_i}{\partial \rho_j} Y_i\right) - C_2^T(\rho)C_2(\rho) & \sum_{i=1}^{N} g_i(\rho)Y_iB_{11}(\rho) & \gamma^{-1}\tilde{C}^T(\rho) \\ B_{11}^T(\rho)\sum_{i=1}^{N} g_i(\rho)Y_i & -I_{n_{d1}} & \gamma^{-1}D_{11\cdot 1}^T(\rho) \\ \gamma^{-1}\tilde{C}(\rho) & \gamma^{-1}D_{11\cdot 1}(\rho) & -I_{n_e} \end{bmatrix},$$

$$Sp^{\infty}(X_1, \cdots, X_N, Y_1, \cdots, Y_N, \rho) := \begin{bmatrix} \sum_{i=1}^{N} f_i(\rho)X_i & \gamma^{-1}I \\ \gamma^{-1}I & \sum_{i=1}^{N} g_i(\rho)Y_i \end{bmatrix}.$$

The next lemma states the convexity of above functions for fixed parameter.

Lemma 4.4.1 *For fixed $\bar{\rho} \in \mathcal{P}$, the scalar valued function $\lambda_{max}\left[Ric_X^{\infty}(X_1, \cdots, X_N, \bar{\rho})\right]$ is convex function of $\{X_i\}_{i=1}^N$, $\lambda_{max}\left[Ric_Y^{\infty}(Y_1, \cdots, Y_N, \bar{\rho})\right]$ is convex function of $\{Y_i\}_{i=1}^N$ and $\lambda_{max}\left[Sp^{\infty}(X_1, \cdots, X_N, Y_1, \cdots, Y_N, \bar{\rho})\right]$ is convex function of $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^N$ jointly.*

Proof: Define $X^1 := \{X_{1,i}\}_{i=1}^N$, $X^2 := \{X_{2,i}\}_{i=1}^N$, $Y^1 := \{Y_{1,i}\}_{i=1}^N$ and $Y^2 := \{Y_{2,i}\}_{i=1}^N$, then

$$\begin{aligned} &\lambda_{max}\left[Sp^{\infty}\left((1-\alpha)X^1 + \alpha X^2, (1-\alpha)Y^1 + \alpha Y^2, \bar{\rho}\right)\right] \\ &= \max_{\substack{x \in \mathbf{R}^n \\ ||x||=1}} x^T \left[Sp^{\infty}\left((1-\alpha)X^1 + \alpha X^2, (1-\alpha)Y^1 + \alpha Y^2, \bar{\rho}\right)\right] x \\ &\leq (1-\alpha) \max_{\substack{x \in \mathbf{R}^n \\ ||x||=1}} x^T \left[Sp^{\infty}(X^1, Y^1, \bar{\rho})\right] x + \alpha \max_{\substack{x \in \mathbf{R}^n \\ ||x||=1}} x^T \left[Sp^{\infty}(X^2, Y^2, \bar{\rho})\right] x \\ &= (1-\alpha)\lambda_{max}\left[Sp^{\infty}(X^1, Y^1, \bar{\rho})\right] + \alpha\lambda_{max}\left[Sp^{\infty}(X^2, Y^2, \bar{\rho})\right] \end{aligned}$$

which clearly shows the convexity of function $\lambda_{max}\left[Sp^{\infty}(X_1, \cdots, X_N, Y_1, \cdots, Y_N, \bar{\rho})\right]$. The convex property of functions $\lambda_{max}\left[Ric_X^{\infty}(X_1, \cdots, X_N, \bar{\rho})\right]$ and $\lambda_{max}\left[Ric_Y^{\infty}(Y_1, \cdots, Y_N, \bar{\rho})\right]$ can be proved similarly without difficulty. ∎

Furthermore, the inequalities in condition (4.4.2.) must hold for all $\rho \in \mathcal{P}$, which include infinite number of constraints to be checked. To solve this infinite constraints convex problem, we generally need to grid the compact set $\mathcal{P}$. For example, if we grid a hyper-rectangle $\mathcal{P} \subset \mathbf{R}^s$ with L points in each dimension, then the convex problem to determine appropriate $\{X_i\}_{i=1}^N$ and $\{Y_i\}_{i=1}^N$ includes approximately $L^s\left(2^{s+1}+1\right)$ LMIs.

The feasibility of these finite number of inequalities can then be determined with techniques in [NesN], [NemG], [BoyE], [NekF], [HaeO].

4.4.2 Complexity Analysis

For feasible functions $X(\rho) = \sum_{i=1}^{N} f_i(\rho) X_i$ and $Y(\rho) = \sum_{i=1}^{N} g_i Y_i$, by compactness of $\mathcal{P}$, we know there exists a small $\delta > 0$, such that for all $\rho \in \mathcal{P}$

$$\begin{aligned} Ric_X^{\infty}(X_1, \cdots, X_N, \rho) &< -\delta, \\ Ric_Y^{\infty}(Y_1, \cdots, Y_N, \rho) &< -\delta, \\ Sp^{\infty}(X_1, \cdots, X_N, Y_1, \cdots, Y_N, \rho) &\geq \delta. \end{aligned}$$

Now we consider the inverse problem of the above. For simplicity, we assume that set $\mathcal{P}$ is a hyper-rectangle in $\mathbf{R}^s$. Gridding $\mathcal{P}$ by $L_1 \times \cdots \times L_s$ points which is uniformly spaced in each dimension, and denote the set of gridding points as

$$\mathcal{V} = \{(\rho_1, \cdots, \rho_s) : \rho_1 \in \{\rho_{1,1}, \cdots, \rho_{1,L_1}\}, \cdots, \rho_s \in \{\rho_{s,1}, \cdots, \rho_{s,L_s}\}\}.$$

Given a large number $T > 0$ and a small number $\delta > 0$, suppose for all $\bar{\rho} \in \mathcal{V}$

$$\|X_i\|_F \leq T, \tag{4.4.3.a}$$

$$\|Y_i\|_F \leq T, \tag{4.4.3.b}$$

$$Ric_X^{\infty}(X_1, \cdots, X_N, \bar{\rho}) \leq -\delta, \tag{4.4.3.c}$$

$$Ric_Y^{\infty}(Y_1, \cdots, X_N, \bar{\rho}) \leq -\delta, \tag{4.4.3.d}$$

$$Sp^{\infty}(X_1, \cdots, X_N, Y_1, \cdots, Y_N, \bar{\rho}) \geq \delta. \tag{4.4.3.e}$$

We want to decide how dense these points should be to guarantee the solvability of the LMIs for all $\rho \in \mathcal{P}$. Note that conditions (4.4.3.a)-(4.4.3.e) are slightly stringent compared with the original ones in equation (4.4.2.).

Lemma 4.4.2 *Given the hyper-rectangle $\mathcal{P} \subset \mathbf{R}^s$, and the LPV system in (4.1.2). Assume that all state-space data are continuously differentiable and f_i, g_i are twice continuously differentiable. Let*

$$h_j^{min} := \left(\frac{\delta}{s}\right) \min \left\{ \left[2T \sum_{i=1}^{N} \max_{\rho \in \mathcal{P}} \left\| \frac{\partial (f_i \hat{A}^T)}{\partial \rho_j} \right\|_F + \nu_j T \sum_{t=1}^{s} \sum_{i=1}^{N} \max_{\rho \in \mathcal{P}} \left| \frac{\partial^2 f_i}{\partial \rho_j \partial \rho_t} \right| \right.\right.$$

$$+\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(B_2B_2^T)}{\partial\rho_j}\right\|_F + 2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(f_iC_{11})}{\partial\rho_j}\right\|_F + 2\gamma^{-1}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial\hat{B}^T}{\partial\rho_j}\right\|_F$$
$$\left. + 2\gamma^{-1}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial D_{111\cdot}^T}{\partial\rho_j}\right\|_F\right]^{-1},$$
$$\left[2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(g_i\tilde{A})}{\partial\rho_j}\right\|_F + \nu_jT\sum_{t=1}^{s}\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{\partial^2 g_i}{\partial\rho_j\partial\rho_t}\right| + \max_{\rho\in\mathcal{P}}\left\|\frac{\partial(C_2^TC_2)}{\partial\rho_j}\right\|_F\right.$$
$$\left. + 2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(g_iB_{11}^T)}{\partial\rho_j}\right\|_F + 2\gamma^{-1}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial\tilde{C}}{\partial\rho_j}\right\|_F + 2\gamma^{-1}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial D_{11\cdot1}^T}{\partial\rho_j}\right\|_F\right]^{-1},$$
$$\left.\left[T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{\partial f_i}{\partial\rho_j}\right| + T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{\partial g_i}{\partial\rho_j}\right|\right]^{-1}\right\}$$

for $j = 1, 2, \cdots, s$. If $|\rho_{j,k_j} - \rho_{j,k_j+1}| \le h_j^{min}$ for all $k_j = 1, 2, \cdots, L_j - 1$ with $j = 1, 2, \cdots, s$, and there exist matrices $\bar{X}_i, \bar{Y}_i$ solving equations (4.4.3.a)-(4.4.3.e), then for all $\rho \in \mathcal{P}$

$$\begin{aligned} Ric_X^{\infty}(\bar{X}_1, \cdots, \bar{X}_N, \rho) &< 0, \\ Ric_Y^{\infty}(\bar{Y}_1, \cdots, \bar{Y}_N, \rho) &< 0, \\ Sp^{\infty}(\bar{X}_1, \cdots, \bar{X}_N, \bar{Y}_1, \cdots, \bar{Y}_N, \rho) &\ge 0. \end{aligned}$$

Proof: Note that for any $\rho \in \mathcal{P}$, there exist some $k_1, \cdots, k_s$ such that $\rho \in [\rho_{1,k_1}, \rho_{1,k_1+1}] \times \cdots \times [\rho_{s,k_s}, \rho_{s,k_s+1}]$. Let $\bar{\rho} := [\rho_{1,k_1} \cdots \rho_{s,k_s}]$, the nonzeros entries of the symmetric matrix $DX(\rho, \bar{\rho}) := Ric_X^{\infty}(X_1, \cdots, X_N, \rho) - Ric_X^{\infty}(X_1, \cdots, X_N, \bar{\rho})$ are

$$\begin{aligned} DX_{11}(\rho,\bar{\rho}) &:= \sum_{i=1}^{N}\left[X_i\left(f_i\hat{A}^T(\rho) - f_i\hat{A}^T(\bar{\rho})\right) + \left(f_i\hat{A}(\rho) - f_i\hat{A}(\bar{\rho})\right)X_i\right] \\ &\quad + \sum_{j=1}^{s}\pm\left[\nu_j\sum_{i=1}^{N}\left(\frac{\partial f_i}{\partial\rho_j}(\rho) - \frac{\partial f_i}{\partial\rho_j}(\bar{\rho})\right)\right] - \left[B_2B_2^T(\rho) - B_2B_2^T(\bar{\rho})\right], \\ DX_{21}(\rho,\bar{\rho}) &:= \left(f_iC_{11}(\rho) - f_iC_{11}(\bar{\rho})\right)X_i, \\ DX_{12}(\rho,\bar{\rho}) &= \left(DX_{\infty}^{(2,1)}(\rho,\bar{\rho})\right)^T \\ DX_{31}(\rho,\bar{\rho}) &:= \gamma^{-1}\left(\hat{B}(\rho) - \hat{B}(\bar{\rho})\right), \\ DX_{13}(\rho,\bar{\rho}) &= \left(DX_{31}(\rho,\bar{\rho})\right)^T \\ DX_{32}(\rho,\bar{\rho}) &:= \gamma^{-1}\left(D_{111\cdot}^T(\rho) - D_{111\cdot}^T(\bar{\rho})\right) \\ DX_{23}(\rho,\bar{\rho}) &= \left(DX_{32}(\rho,\bar{\rho})\right)^T. \end{aligned}$$

So,

$\|DX(\rho,\bar{\rho})\|_F$

$$
\begin{aligned}
\leq\ & 2\sum_{i=1}^{N}\left\|X_i\left(f_i\hat{A}^T(\rho)-f_i\hat{A}^T(\bar{\rho})\right)\right\|_F+\sum_{j=1}^{s}\left[\nu_j\sum_{i=1}^{N}\left\|X_i\left(\frac{\partial f_i}{\partial\rho_j}(\rho)-\frac{\partial f_i}{\partial\rho_j}(\bar{\rho})\right)\right\|_F\right] \\
& +\left\|B_2B_2^T(\rho)-B_2B_2^T(\bar{\rho})\right\|_F+2\sum_{i=1}^{N}\left\|X_i\left(f_iC_{11}(\rho)-f_iC_{11}(\bar{\rho})\right)\right\|_F \\
& +2\gamma^{-1}\left\|\hat{B}^T(\rho)-\hat{B}^T(\bar{\rho})\right\|_F+2\gamma^{-1}\left\|D_{111\cdot}^T(\rho)-D_{111\cdot}^T(\bar{\rho})\right\|_F \\
\leq\ & \sum_{j=1}^{s}|\rho_j-\rho_{j,k_j}|\left[2\sum_{i=1}^{N}\left\|X_i\frac{\partial(f_i\hat{A}^T)}{\partial\rho_j}(\xi^{1ij})\right\|_F+\nu_j\sum_{t=1}^{s}\sum_{i=1}^{N}\left\|X_i\frac{\partial^2 f_i}{\partial\rho_j\rho_t}(\xi^{2ijt})\right\|_F\right. \\
& +\left\|\frac{\partial(B_2B_2^T)}{\partial\rho_j}(\xi^{3j})\right\|_F+2\sum_{i=1}^{N}\left\|X_i\frac{\partial(f_iC_{11})}{\partial\rho_j}(\xi^{4ij})\right\|_F+2\gamma^{-1}\left\|\frac{\partial\hat{B}^T}{\partial\rho_j}(\xi^{5j})\right\|_F \\
& \left.+2\gamma^{-1}\left\|\frac{\partial D_{111\cdot}^T}{\partial\rho_j}(\xi^{6j})\right\|_F\right], \\
& \qquad \text{for some } \xi^{1ij},\xi^{2ijt},\xi^{3j},\xi^{4ij},\xi^{5j},\xi^{6j}\in[\rho_{1,k_1},\rho_{1,k_1+1}]\times\cdots[\rho_{s,k_s},\rho_{s,k_s+1}] \\
\leq\ & \sum_{j=1}^{s}h_j^{min}\left[2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(f_i\hat{A}^T)}{\partial\rho_j}\right\|_F+\nu_jT\sum_{t=1}^{s}\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{\partial^2 f_i}{\partial\rho_j\rho_t}\right|+\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(B_2B_2^T)}{\partial\rho_j}\right\|_F\right. \\
& \left.+2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial(f_iC_{11})}{\partial\rho_j}\right\|_F+2\gamma^{-1}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial\hat{B}^T}{\partial\rho_j}\right\|_F+2\gamma^{-1}\max_{\rho\in\mathcal{P}}\left\|\frac{\partial D_{111\cdot}^T}{\partial\rho_j}\right\|_F\right] \\
\leq\ & \delta.
\end{aligned}
$$

Therefore, $\bar{\sigma}\left[Ric_X^\infty(X_1,\cdots,X_N,\rho)-Ric_X^\infty(X_1,\cdots,X_N,\bar{\rho})\right]\leq\delta$, which guarantees

$$Ric_X^\infty(X_1,\cdots,X_N,\rho)<0$$

for all $\rho\in\mathcal{P}$ as desired. The other two terms can be derived similarly. ∎

People may complain about the need to grid the $\mathcal{P}$ set. We too feel that this is a disadvantage of the method. However in many gain-scheduling applications, the number of scheduling variables is small, usually 3 or less. Hence the dimensionality of the gridding, while extremely cumbersome, is not overwhelming. Of course, for a problem with many parameters, the gridding procedure will become prohibitively expensive. This clearly indicates the drawbacks associated with using Theorem 3.3.1 as a general robustness analysis tool for systems with time-varying real uncertainty.

Another significant problem is the complete lack of guidance provided by the theory to pick the basis functions, namely, f_i and g_i. An intuitive rule for basis function selection is to use those present in the open-loop state-space data. Hopefully future study will yield some results along these lines.

Chapter 5

Parameter-Dependent Stabilization of LPV Systems

In this chapter, the problem of stabilizing LPV systems using parameter-dependent Lyapunov functions is studied and the parameterization of all parametrically-dependent stabilizing controllers is given explicitly. We consider the case where parameter's derivative is also measurable in real-time. The results can be extended to the case that both plant and controller depend on parameter only. It will be interesting to formulate a general framework for such problem but we are not pursuing here. This chapter is a straightforward generalization of the problem studied in [PacB], [Bec], and we recover their results by allowing arbitrarily fast varying parameters and restricting constant Lyapunov functions. All results are easily derived with slight modification of the corresponding ones in [Bec], thus proofs are omitted here.

Specifically, in §5.1 we state parametrically-dependent stabilization problem for LPV systems. §5.2 is devoted to the discussion of parameter-dependent stabilizability, parameter-dependent detectability, and their properties. In §5.3, we solve the parametrically-dependent stabilization problem by parameterizing the set of all parametrically-dependent stabilizing controllers.

5.1 Parameter-Dependent Stabilization Problem

In this section we define a parametrically-dependent output-feedback stabilization problem for LPV systems. This problem is a generalization of quadratic LPV stabilization problem

[PacB], [Bec] but using PDLF. The open-loop LPV system is in the form of

Definition 5.1.1 Open-Loop LPV System for Parameter-Dependent Stabilization

Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, the n-th order open-loop LPV system is given by

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_2(\rho(t)) \\ C_2(\rho(t)) & 0_{n_y \times n_u} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \tag{5.1.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^{n_u}$, and $y(t) \in \mathbf{R}^{n_y}$, and the state-space data is of appropriate dimension.

Note that the "D"term (direct feed-through from u to y) is assumed to be zero in equation (5.1.1). It can be relaxed and the controller's formulae is easily fixed at the expense of complexity (see [Bec]). The subscript 2 of matrix functions B_2 and C_2 is used to distinguish u and y from exogenous disturbance inputs and error outputs which are included later on for performance problem.

The class of parameter-dependent controller we consider is assumed to depend on parameter and its derivative, and it is defined clearly in the following:

Definition 5.1.2 Parameter-Dependent Controller

Given a compact set $\mathcal{P} \in \mathbf{R}^s$, an integer $m \geq 0$, and the continuous functions (A_K, B_K, C_K, D_K) : $\mathbf{R}^s \times \mathbf{R}^s \to (\mathbf{R}^{m\times m}, \mathbf{R}^{m\times n_y}, \mathbf{R}^{n_u \times m}, \mathbf{R}^{n_u \times n_y})$. Then the parameter-dependent linear feedback controller can be written as

$$\begin{bmatrix} \dot{x}_k(t \\ u(t) \end{bmatrix} = \begin{bmatrix} A_K(\rho(t),\dot{\rho}(t)) & B_K(\rho(t),\dot{\rho}(t)) \\ C_K(\rho(t),\dot{\rho}(t)) & D_K(\rho(t),\dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x_k(t) \\ y(t) \end{bmatrix}, \tag{5.1.2}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, x_k is the m-dimensional controller states. This controller is denoted by $K_{\mathcal{P}}^m$ (or simply $K_{\mathcal{P}}$ when the dimension is clear).

Defining $x_{\text{clp}}^T := \begin{bmatrix} x^T & x_k^T \end{bmatrix}$, and applying the controller $K_{\mathcal{P}}$ (5.1.2) to the LPV system (5.1.1), the resulting closed-loop LPV system is given by

$$\dot{x}_{\text{clp}}(t) = A_{\text{clp}}(\rho(t),\dot{\rho}(t))x_{\text{clp}}(t)$$

where

$$A_{\text{clp}}(\rho,\dot{\rho}) = \begin{bmatrix} A(\rho) + B_2(\rho)D_K(\rho,\dot{\rho})C_2(\rho) & B_2(\rho)C_K(\rho,\dot{\rho}) \\ B_K(\rho,\dot{\rho})C_2(\rho) & A_K(\rho,\dot{\rho}) \end{bmatrix}. \tag{5.1.3}$$

For notational purposes, define following functions,

$$A_m^e(\rho) := \begin{bmatrix} A(\rho) & 0 \\ 0 & 0_m \end{bmatrix} \quad B_m^e(\rho) := \begin{bmatrix} 0 & B_2(\rho) \\ I_m & 0 \end{bmatrix}$$

$$C_m^e(\rho) := \begin{bmatrix} 0 & I_m \\ C_2(\rho) & 0 \end{bmatrix} \quad K(\rho,\dot\rho) := \begin{bmatrix} A_K(\rho,\dot\rho) & B_K(\rho,\dot\rho) \\ C_K(\rho,\dot\rho) & D_K(\rho,\dot\rho) \end{bmatrix}.$$

Then it is easy to check that A_{clp} in (5.1.3) can be rewritten as $A_{\mathrm{clp}}(\rho,\dot\rho) = A_m^e(\rho) + B_m^e(\rho)K(\rho,\dot\rho)C_m^e(\rho)$.

The parameter-dependent stabilization problem asks for parameter-dependent stability for closed-loop systems and is stated as follows:

Definition 5.1.3 Parameter-Dependent Stabilization Problem
Given an open-loop LPV system in Definition 1.2.2. The Parameter-Dependent Stabilization Problem is solvable if there exist an integer $m \ge 0$, a function $Z \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{(n+m)\times(n+m)})$, and a function $K \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{(n_u+m)\times(n_y+m)})$, such that $Z(\rho) > 0$ and

$$\left[A_m^e(\rho) + B_m^e(\rho)K(\rho,\beta)C_m^e(\rho)\right]^T Z(\rho) + Z(\rho)\left[A_m^e(\rho) + B_m^e(\rho)K(\rho,\beta)C_m^e(\rho)\right] + \sum_{i=1}^{s}\left(\beta_i \frac{\partial Z}{\partial \rho_i}\right) < 0$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i,\ i = 1,\cdots,s$.

Later in §5.3, we will show that the solvability of the parameter-dependent stabilization is based on an associated parameter-dependent state-feedback stabilization and observation problems are solvable.

5.2 Parameter-Dependent Stabilizability and Detectability

In this section, we introduce the concepts of parameter-dependent stabilizability and parameter-dependent detectability, and their properties. These concepts are generalizations of quadratic stabilizability and quadratic detectability (see [Bec]) by using PDLF.

5.2.1 Parameter-Dependent Stabilizability

We state the notation of parameter-dependent stabilizability and its equivalent condition. Also we show that adding dynamics to state-feedback controller does not help to stabilize LPV systems.

Definition 5.2.1 Parameter-Dependent Stabilizability

The pair of matrix functions (A, B_2) *are parametrically-dependent stabilizable (PDS) over* $\mathcal{P}$ *if there exist functions* $P_F \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $F \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n_u\times n})$ *such that,* $P_F(\rho) > 0$ *and*

$$P_F(\rho)\left[A(\rho) + B_2(\rho)F(\rho,\beta)\right] + \left[A(\rho) + B_2(\rho)F(\rho,\beta)\right]^T P_F(\rho) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P_F}{\partial \rho_i}\right) < 0$$

for all $\rho \in \mathcal{P}$ *and* $|\beta_i| \le \nu_i$, $i = 1, \cdots, s$. *Such a function* F *will be referred to as a parametrically-dependent stabilizing state-feedback gain for the pair* (A, B_2) *over* $\mathcal{P}$.

By applying the parameter-dependent state-feedback control $u := F(\rho, \dot{\rho})x$ to the open-loop LPV system

$$\dot{x}(t) = A(\rho(t))x(t) + B_2(\rho(t))u(t),$$

the resulting closed-loop system is parametrically-dependent stable for any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. The stability can be shown by PDLF $V(x,\rho) := x^T P_F(\rho)x$ (by Definition 3.2.3).

The following lemma gives an equivalent conditions to Definition 5.2.1. It has computational advantages over Definition 5.2.1 because unknown variables in the condition (5.2.1) are shown in the affine form.

Lemma 5.2.1 *The pair* (A, B_2) *is PDS over* $\mathcal{P}$ *if and only if there exist functions* $W \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $R \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n_u\times n})$ *such that* $W(\rho) > 0$ *and*

$$A(\rho)W(\rho) + W(\rho)A^T(\rho) + B_2(\rho)R(\rho,\beta) + R^T(\rho,\beta)B_2^T(\rho) - \sum_{i=1}^{s}\left(\beta_i \frac{\partial W}{\partial \rho_i}\right) < 0 \quad (5.2.1)$$

for $\rho \in \mathcal{P}$ *and* $|\beta_i| \le \nu_i, i = 1, \cdots, s$.

The next lemma shows that including dynamics in the state-feedback controller does not assist in the parameter-dependent stabilization. This lemma is a special case of more general results found in [RotK1] and [RotK2].

Lemma 5.2.2 *Let* m *be any non-negative integer. Then the pair* (A, B_2) *is PDS over* $\mathcal{P}$ *if and only if the pair* (A_m^e, B_m^e) *is PDS over* $\mathcal{P}$.

5.2.2 Parameter-Dependent Detectability

Similarly, we can define parameter-dependent detectability, which is the dual of parameter-dependent stabilizability.

Definition 5.2.2 Parameter-Dependent Detectability
The pair of matrix functions (A, C_2) *is parametrically-dependent detectable (PDD) over* $\mathcal{P}$ *if there exist functions* $P_L \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $L \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n\times n_y})$ *such that* $P_L(\rho) > 0$ *and*

$$P_L(\rho)\left[A(\rho) + L(\rho,\beta)C_2(\rho)\right] + \left[A(\rho) + L(\rho,\beta)C_2(\rho)\right]^T P_L(\rho) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P_L}{\partial \rho_i}\right) < 0$$

for all $\rho \in \mathcal{P}$ *and* $|\beta_i| \leq \nu_i, i = 1, \ldots, s$. *Such a function* L *will be referred to as a parametrically-dependent output injection gain for the pair* (A, C_2) *over* $\mathcal{P}$.

Similar to parameter-dependent stabilizability, using such a L in the parametrically-dependent observer

$$\dot{\hat{x}}(t) = A(\rho(t))\hat{x}(t) - L(\rho(t), \dot{\rho}(t))\left[y(t) - C(\rho(t))\hat{x}(t)\right]$$

yields a parametrically-dependent stable estimation of states x for the LPV system

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) \\ y(t) &= C(\rho(t))x(t). \end{aligned}$$

for any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$.

The equivalent condition of parameter-dependent detectability is the following:

Lemma 5.2.3 *The pair* (A, C_2) *is PDD if and only if there exist functions* $P \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ *and* $H \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n\times n_y})$ *such that* $P(\rho) > 0$ *and*

$$A^T(\rho)P(\rho) + P(\rho)A(\rho) + H(\rho,\beta)C_2(\rho) + C_2^T(\rho)H^T(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P}{\partial \rho_i}\right) < 0 \quad (5.2.2)$$

for all $\rho \in \mathcal{P}$ *and* $|\beta_i| \leq \nu_i, i = 1, \ldots, s$.

As in the state-feedback stabilizability problem, dynamic extension does not make the observation problem any easier also.

Lemma 5.2.4 *Let* m *be any nonnegative integer. Then, the pair* (A, C_2) *is PDD over* $\mathcal{P}$ *if and only if the pair* (A_m^e, C_m^e) *is PDD over* $\mathcal{P}$.

5.3 Parameter-Dependent Stabilization: Controller Synthesis and Parametrization

In this section we formulate necessary and sufficient conditions for the solution to the Parameter-Dependent Stabilization Problem. It turns out that the problem using output-feedback control is solvable if and only if the associated state-feedback and output injection problems are solvable. The theorem states:

Theorem 5.3.1 *Given the LPV system in Definition 1.2.2, the Parameter-dependent Stabilization Problem is solvable if and only if the following two conditions hold:*

1. *there exist functions $P_F \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$ and $F \in \mathcal{C}^0(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n_u \times n})$ such that for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i,\ i = 1, \cdots, s$, $P_F(\rho) > 0$ and*
$$[A(\rho) + B_2(\rho)F(\rho,\beta)]^T P_F(\rho) + P_F(\rho)\,[A(\rho) + B_2(\rho)F(\rho,\beta)] + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P_F}{\partial \rho_i}\right) < 0, \tag{5.3.1}$$

2. *there exist functions $P_L \in \mathcal{C}^1(\mathbf{R}^s, \mathcal{S}^{n\times n})$, and $L \in \mathcal{C}^1(\mathbf{R}^s \times \mathbf{R}^s, \mathbf{R}^{n \times n_y})$ such that for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i,\ i = 1, \cdots, s$,*
$$[A(\rho) + L(\rho,\beta)C_2(\rho)]^T P_L(\rho) + P_L(\rho)\,[A(\rho) + L(\rho,\beta)C_2(\rho)] + \sum_{i=1}^{s}\left(\beta_i \frac{\partial P_L}{\partial \rho_i}\right) < 0. \tag{5.3.2}$$

If the functions F and L exist as in the above conditions, then

$$K(\rho,\dot{\rho}) := \begin{bmatrix} A(\rho) + B_2(\rho)F(\rho,\dot{\rho}) + L(\rho,\dot{\rho})C_2(\rho) & F(\rho,\dot{\rho}) \\ -L(\rho,\dot{\rho}) & 0 \end{bmatrix}$$

is the state-space data of one parametrically-dependent stabilizing output-feedback controller.

Theorem 5.3.1 gives one particular parametrically-dependent stabilizing output-feedback controller. Now we will parameterize all parametrically-dependent stabilizing controllers for the LPV systems in (5.1.1). The following theorem has the same parameterization as [Bec, Theorem 4.2.10], which is the familiar observer/state-feedback/stable-operator structure that is well known for LTI and LTV systems.

Theorem 5.3.2 *Given the open-loop LPV system in Definition 1.2.2, such that the pair* (A, B_2) *is PDS over* $\mathcal{P}$ *using some state-feedback* $F(\rho, \dot{\rho})$ *and* (A, C_2) *is PDD over* $\mathcal{P}$ *using some output injection* $L(\rho, \dot{\rho})$*. Then the input/output behavior of all linear parameter varying output-feedback controllers achieving parameter-dependent stability over* $\mathcal{P}$ *is parametrized as*

$$\begin{bmatrix} \dot{\eta}_{\mathrm{os}}(t) \\ \dot{\eta}_{\mathrm{Q}}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_{K11}(\rho(t), \dot{\rho}(t)) & A_{K12}(\rho(t), \dot{\rho}(t)) & B_{K1}(\rho(t), \dot{\rho}(t)) \\ A_{K21}(\rho(t), \dot{\rho}(t)) & A_Q(\rho(t), \dot{\rho}(t)) & -B_Q(\rho(t), \dot{\rho}(t)) \\ C_{K1}(\rho(t), \dot{\rho}(t)) & C_Q(\rho(t), \dot{\rho}(t)) & -D_Q(\rho(t), \dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} \eta_{\mathrm{os}}(t) \\ \eta_{\mathrm{Q}}(t) \\ y(t) \end{bmatrix}, \qquad (5.3.3)$$

where

$$\begin{aligned} A_{K11}(\rho, \dot{\rho}) &:= A(\rho) + B_2(\rho)F(\rho, \dot{\rho}) + L(\rho, \dot{\rho})C_2(\rho) + B_2(\rho)D_Q(\rho, \dot{\rho})C_2(\rho) \\ A_{K12}(\rho, \dot{\rho}) &:= B_2(\rho)C_Q(\rho, \dot{\rho}), \qquad A_{K21}(\rho, \dot{\rho}) := B_Q(\rho, \dot{\rho})C_2(\rho) \\ B_{K1}(\rho, \dot{\rho}) &:= -L(\rho, \dot{\rho}) + B_2(\rho)D_Q(\rho, \dot{\rho}), \qquad C_{K1}(\rho, \dot{\rho}) := F(\rho, \dot{\rho}) + D_Q(\rho, \dot{\rho})C_2(\rho), \end{aligned}$$

and the matrices A_Q, B_Q, C_Q *and* D_Q *are arbitrary continuous functions (of appropriate dimensions) on* $\mathbf{R}^s \times \mathbf{R}^s$*, with* A_Q *parametrically-dependent stable over* $\mathcal{P}$*.*

A realization of this familiar controller is shown in Figure 5.1.

The controller's formula in equation (5.3.3) can also be interpreted with the usual linear fractional transformation. Let F and L be the parametrically-dependent stabilizing gain for the pairs (A, B_2) and (A, C_2) as in Definitions 5.3.1 and 5.3.2. Define the 2-input, 2-output parametrically dependent operator J_ρ as

$$J_\rho(\rho, \dot{\rho}) := \left[\begin{array}{c|cc} A(\rho) + B_2(\rho)F(\rho, \dot{\rho}) + L(\rho, \dot{\rho})C_2(\rho) & -L(\rho, \dot{\rho}) & B_2(\rho) \\ \hline F(\rho, \dot{\rho}) & 0 & I \\ C_2(\rho) & -I & 0 \end{array} \right]. \qquad (5.3.4)$$

Then, all FDLTV output-feedback, parametrically dependent controllers achieving parameter-dependent stability over $\mathcal{P}$ are parametrized as the interconnection block diagram in Figure 5.2, where Q_ρ is any parametrically-dependent stable LPV system over $\mathcal{P}$.

Suppose the state-space data of this LPV system Σ_ρ is written as

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & D_{11}(\rho(t)) & D_{12}(\rho(t)) \\ C_2(\rho(t)) & D_{21}(\rho(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}. \qquad (5.3.5)$$

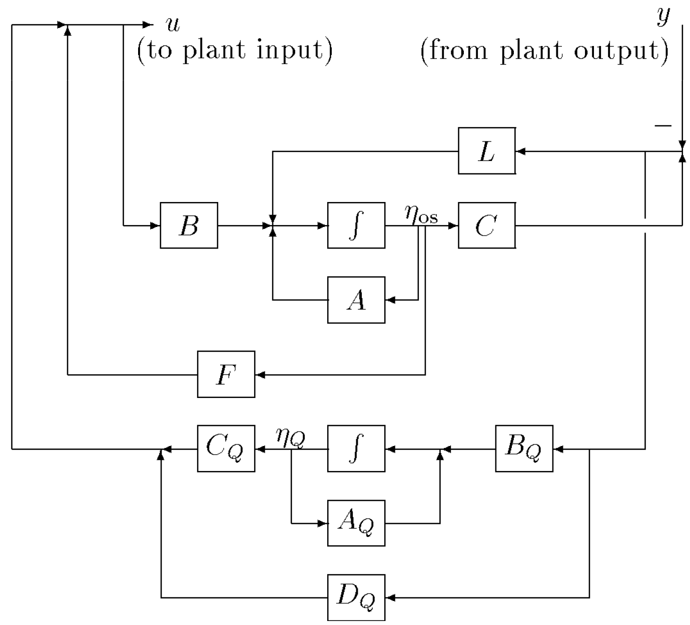

Figure 5.1: Parametrization of all finite-dimensional linear time-varying parametrically dependent controllers achieving parameter-dependent stability.

Define the 2-input, 2-output parametrically dependent operator T_ρ as

$$T_\rho = \left[\begin{array}{cc|cc} A & B_2F & B_1 & B_2 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} & B_2 \\ \hline C_1 & D_{12}F & D_{11} & D_{12} \\ -C_2 & C_2 & D_{21} & 0 \end{array}\right]. \tag{5.3.6}$$

Then, for all parametrically-dependent stable operators Q_ρ, with state-space realization

$$Q_\rho(\rho, \dot\rho) = \left[\begin{array}{c|c} A_Q(\rho, \dot\rho) & B_Q(\rho, \dot\rho) \\ \hline C_Q(\rho, \dot\rho) & D_Q(\rho, \dot\rho) \end{array}\right], \tag{5.3.7}$$

the block diagrams in Figure 5.3 are zero-state equivalent, and they represent all of the possible closed-loop disturbance-to-error operators using parametrically-dependent stabilizing controllers. For all parametrically-dependent stabilizing controllers, the zero-state, closed-loop operator from d to e is an affine function of the free parameter Q because the $(2, 2)$ entry of T is zero.

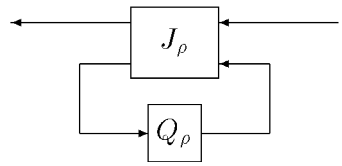

Figure 5.2: $J_\rho - Q_\rho$ parametrization of all FDLTV output-feedback, parametrically dependent controllers achieving parameter-dependent stability.

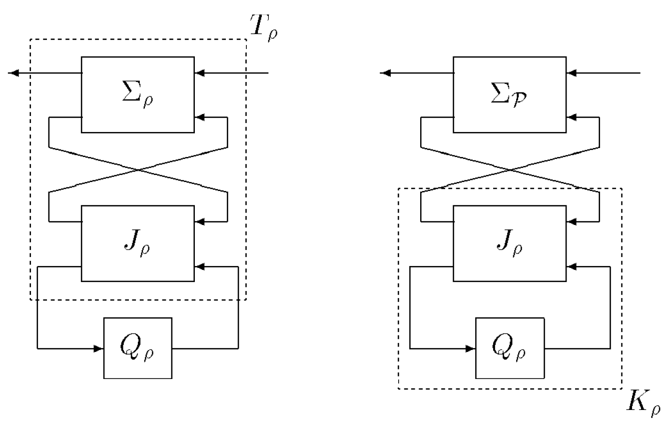

Figure 5.3: $\Sigma_\rho - J_\rho - Q_\rho$ interconnection

Chapter 6

LPV Systems Controller Design

In this chapter, we study several examples which illustrate the usefulness of the LPV system theory to the gain-scheduling design.

Specifically, in §6.1 we design a controller with guaranteed LQG performance for a two rotating disks problem. In §6.2, the design of LPV control with induced $\mathbf{L}_2$-norm performance for a pitch-axis missile problem is considered. In §6.3, we use the two-disk problem introduced in §6.2 as a benchmark to compare the performance of different control design methods.

6.1 LQG Control Example

In this section, we will design the LPV controller with guaranteed LQG performance for a two-disk control problem. The problem is artificial and introduced in [Bec]. It is used here to show the design of LPV controllers.

6.1.1 Two-Disk Model and Performance Measure

The problem is to control the radial position of a slider (M_2) on a rod which could rotate in the horizontal plane. The slider to be positioned is coupled with another slider (M_1) mounted on a separate rotating rod. The control action applied to M_1 is transmitted to M_2 through a wire which can transmit force in both compression and tension. The sliders are free to slide along the rods which rotate in the horizontal plane with angular velocity Ω_1 and Ω_2 respectively.

The dynamics of this two-disk system are represented by the following equations:

$$M_1\left[\ddot{r}_1(t) - \Omega_1^2(t)r_1(t)\right] = -b\dot{r}_1(t) - k\left[r_1(t) + r_2(t)\right] + f(t) \quad (6.1.1)$$

$$M_2\left[\ddot{r}_2(t) - \Omega_2^2(t)r_2(t)\right] = -b\dot{r}_2(t) - k\left[r_1(t) + r_2(t)\right] \quad (6.1.2)$$

The various variables in above equations are

$r_1(t)$	position of the first slider relative to the center (m)
$r_2(t)$	position of the second slider relative to the center (m)
$\Omega_1(t)$	rotational rate of the first rod varying between 0 and 3 rad/sec
$\Omega_2(t)$	rotational rate of the second rod varying between 0 and 5 rad/sec
$f(t)$	control force on the first slider along slot (N).

and the constant values are listed in Table 6.1.

$M_1 = 1.0\ kg$	mass of the first slider
$M_2 = 0.5\ kg$	mass of the second slider
$b = 1.0\ kg/sec$	damping coefficient in slots
$k = 200.0\ N/m$	spring constant

Table 6.1: Coefficients of two-disk problem

We assume that sensors measure the radial position $r_2(t)$ of the slider M_2 and the angular velocities of the rods $\Omega_1(t)$ and $\Omega_2(t)$. Define $\rho_1 := \Omega_1^2,\ \rho_2 := \Omega_2^2$, then $\rho_1(t) \in [0,\ 9]$ and $\rho_2(t) \in [0,\ 25]$. Let $x_1 := r_1$, $x_2 := r_2$, $x_3 := \dot{r}_1$, $x_4 := \dot{r}_2$, $u := f$ and $y := r_2$, then the system $T_{\mathcal{P}}$ of plant model (6.1.1)-(6.1.2) can be written in the LPV form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \rho_1(t) - \frac{k}{M_1} & -\frac{k}{M_1} & -\frac{b}{M_1} & 0 & \frac{0.1}{M_1} & \frac{1}{M_1} & 0 \\ -\frac{k}{M_2} & \rho_2(t) - \frac{k}{M_2} & 0 & -\frac{b}{M_2} & 0 & 0 & \frac{0.1}{M_2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ u(t) \\ d_1(t) \\ d_2(t) \end{bmatrix}, \quad (6.1.3)$$

where $\mathcal{P} := [0,\ 9] \times [0,\ 25]$.

The control objective is: to minimize the effect of reference input, measurement noise and disturbances on the tracking error with respect to "small" commanded input. Based on the time-invariant ideas, these objectives are quantified by rational weighting functions and the weighted open-loop interconnection is given in Figure 6.1.

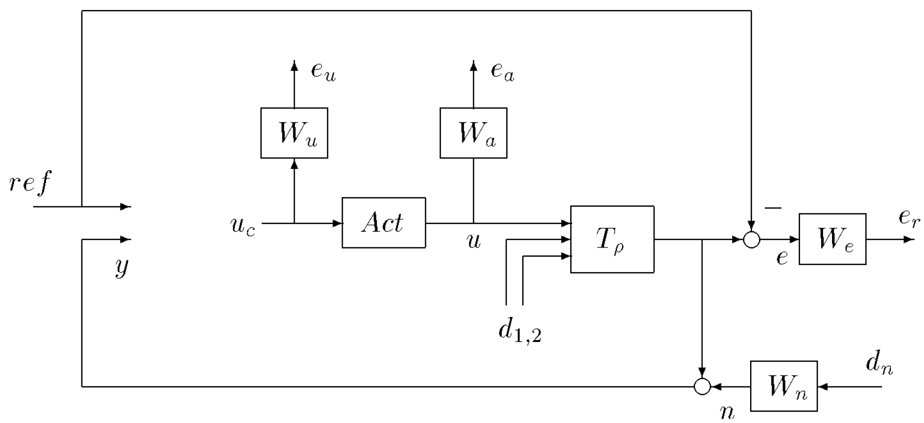

Figure 6.1: Weighted open-loop plant interconnection of two-disk problem.

where

$$\begin{aligned} W_e(s) &:= \frac{2}{s+0.04}, \\ W_u(s) &:= \frac{1}{50}, \\ W_a(s) &:= 0.00001, \\ W_n(s) &:= \frac{s+0.4}{0.01s+400}, \\ Act(s) &:= \frac{1}{0.01s+1}. \end{aligned}$$

With this setup, the LQG performance measure to be minimized is given by

$$\sigma_\infty = \lim_{T\to\infty} \sigma_T,$$

with $\sigma_T = \mathcal{E}\left\{\frac{1}{T}\int_0^T \left[e_r^T(t)e_r(t) + e_u^T(t)e_u(t) + e_a^T(t)e_a(t)\right] dt\right\}$.

6.1.2 Synthesis and Simulation Results

The weighted open-loop system has 7 states, 5 outputs and 5 inputs. Because the state-space data of LPV system (6.1.3) are affine functions of parameter, and B_2, C_2 are constants, it is sufficient to do the synthesis only for all "corners" of the rectangular parameter space, namely, $\{(0,0),(9,0),(0,25),(9,25)\}$. Follow the "one-step" scheme in Theorem 2.5.2 and note that Remark 2.5.1 holds for this problem, we solve for X, Y and obtain $\sigma_\infty \leq \gamma_{sub} = 2.54$ using linear objective solver (LINOBJ) in LMIlab [GahNLC]. This implies that for white noise command input ref and disturbances d_1, d_2 with unit intensity, the expectation of the integral $\int_0^T \left[e_r^T(t)e_r(t) + e_u^T(t)e_u(t) + e_a^T(t)e_a(t)\right] dt$ averaged over time interval $[0,\ T]$ is less than 2.54 for all $T > 0$ and $\rho \in \mathcal{F_P}$.

With synthesized X, Y, we can construct output-feedback LPV controller directly. The state-feedback LPV controller plus Kalman filter is resulted by real-time implementation of the Kalman filter. Then we want to evaluate the performance of two LPV controllers from different aspects.

LQG Performance for Frozen Parameter

For each fixed point in the set $\mathcal{P}$, we can synthesize the $\mathcal{H}_2$ optimal controller and compute its LQG performance. Also we can calculate the LQG performance of closed-loop systems using two LPV controllers evaluated at the same fixed parameter. The frozen performance at 9 fixed points for both cases are shown in Tables 6.2 and Table 6.3 respectively.

		Ω_2		
		0.0	3.5355	5.0
	0.0	2.0185	2.1003	2.2061
Ω_1	2.1213	2.0792	2.1758	2.2931
	3.0	2.1528	2.2617	2.3883

Table 6.2: Frozen $\mathcal{H}_2$ optimal closed-loop LQG performance.

		Ω_2		
		0.0	3.5355	5.0
	0.0	2.2610/2.2556	2.2253/2.2206	2.2445/2.2409
Ω_1	2.1213	2.2461/2.2412	2.2430/2.2391	2.3063/2.3038
	3.0	2.2522/2.2480	2.2882/2.2853	2.4057/2.4040

Table 6.3: Closed-loop LQG performance using output-feedback LPV controller/state-feedback LPV controller plus Kalman filter at frozen parameters.

From above tables, the optimal LQG performance at these 9 points ranges from 2.02 to 2.39. The performance using output-feedback LPV control is between 2.23 to 2.41, and state-feedback LPV control plus Kalman filter is from 2.22 to 2.40. The closed-loop performances of both LPV controllers are very close while the state-feedback LPV control plus Kalman filter performs slightly better. Both of them are about 0.7% to 12% higher than $\mathcal{H}_2$ optimal controller. LPV controls are inferior to $\mathcal{H}_2$ optimal control at fixed parameters because they are designed with respect to time-varying parameters. The maximum of LQG performance using both LPV controllers is 2.41, which is less than the guaranteed performance bound 2.54 as expected.

Controller Frequency Response

We then compare the frequency responses of $\mathcal{H}_2$ optimal controller and two LPV controllers at fixed parameters. All of the controllers have two inputs (reference input ref and measurement y) and one output (control input u_c). The frequency responses from ref to u_c for 9 fixed points are shown in Figures 6.4, 6.5, 6.6, 6.7, 6.8, 6.9. Similarly, the frequency responses of the second channel (from y to u_c) are given by Figures 6.10, 6.11, 6.12, 6.13, 6.14, 6.15.

Although two LPV controllers achieve almost identical LQG performance for fixed parameter, and have similar time response as shown later, we observe that the frequency response of state-feedback control plus Kalman filter is similar to that of the optimal controller, but different from that of the output-feedback LPV controller.

Step Response

Next we compare the fixed parameter closed-loop performance of $\mathcal{H}_2$ optimal controller and two LPV controllers while tracking an unit step reference command. The step responses at 9 fixed parameters are shown in Figures 6.16, 6.17, 6.18. The corresponding control forces are given by Figures 6.19, 6.20, 6.21. Note that the response using state-feedback controller plus Kalman filter coincides with the one using output-feedback controller.

Again, we observe that LPV controlled step responses and control efforts are very close. The tracking performances (rise time, tracking error) of closed-loop system using two LPV controllers are slightly inferior to the optimal case, and control forces are also higher, which shows the sub-optimality of the LPV controllers for fixed parameters.

After carefully comparing the characteristics of LPV controllers with $\mathcal{H}_2$ optimal controller for fixed parameters, we will study the behavior of LPV systems with time-varying parameter trajectory by nonlinear simulation.

Simulation of LPV System with Time-Varying Trajectory

The parameter trajectories are chosen to be

$$\rho_1(t) = 4.5\,(\sin(1.5t) + 1)\,, \qquad \rho_2(t) = 12.5\,(\cos(0.7t) + 1)\,.$$

for simulation purpose. The reference input is a sequence of step commands, and disturbances, noise are set to zeros. For comparison, we can also simulate the response of $\mathcal{H}_2$ optimal LTV control for a given parameter trajectory. The step responses and actual control forces for such parameter variations using output-feedback LPV controller, state-feedback LPV controller plus Kalman filter and optimal LTV controller are shown in Figures 6.22, 6.23. Note that the reference command is in dash-dot line.

From the simulation results, we clearly see that the behaviors of both LPV controllers are quite similar, which was observed before in fixed parameter case. Furthermore, their performances are comparable with $\mathcal{H}_2$ optimal LTV controller. Note that the optimal LTV controller can only be synthesized with respect to a specific parameter trajectory before hand, while LPV controllers are constructed without a priori information of parameters. But output-feedback LPV controller is more suitable for this problem because its implementation does not require real-time computation of differential Riccati equation related to Kalman filter.

6.2 Induced L_2-Norm Control Example

In this section, we design an LPV controller for a missile pitch-axis autopilot. Our method is different from the approach proposed in [NicRR], where traditional gain-scheduling method is combined with extended linearization ideas [BauR], [Rug]. They synthesize linear time-invariant $\mathcal{H}_\infty$ controller at distinct operating points, and interpolate these controllers with respect to the operating condition of the missile. As pointed out in their paper, the gain-scheduling controller should perform well for small, sufficiently slow varying signals. Evaluation of control performance in more demanding situations essentially requires extensive simulations. In LPV control approach, the missile is treated as a single entity, and the gain-scheduling is achieved entirely by the parameter-dependent controller. The closed-loop system is guaranteed with quadratic stability and bounded induced-$\mathcal{L}_2$-norm performance as long as the parameter stays in the given bounded set. The systematic scheme of LPV control theory largely simplifies the design procedure, and provides theoretical justification for the gain-scheduling design.

6.2.1 Missile Model and Performance Objective

The pitch-axis missile model is described by

$$\dot{\alpha}(t) = K_\alpha M(t) C_n \left[\alpha(t), \delta(t), M(t)\right] cos(\alpha(t)) + q(t) \tag{6.2.1}$$

$$\dot{q}(t) = K_q M^2(t) C_m \left[\alpha(t), \delta(t), M(t)\right], \tag{6.2.2}$$

where the aerodynamic coefficients are

$$C_n\left[\alpha, \delta, M\right] = sgn(\alpha)\left[a_n|\alpha|^3 + b_n|\alpha|^2 + c_n\left(2 - \frac{M}{3}\right)|\alpha|\right] + d_n\delta$$

$$C_m\left[\alpha, \delta, M\right] = sgn(\alpha)\left[a_m|\alpha|^3 + b_m|\alpha|^2 + c_m\left(-7 + \frac{8M}{3}\right)|\alpha|\right] + d_m\delta,$$

and the output is normal acceleration

$$\eta(t) = K_z M^2(t) C_n\left[\alpha(t), \delta(t), M(t)\right]. \tag{6.2.3}$$

Actuator dynamics describing the tail deflection are

$$\frac{d}{dt}\begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta\omega_a \end{bmatrix}\begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_a^2 \end{bmatrix}\delta_c(t), \tag{6.2.4}$$

The various variables in plant model are

$\alpha(t)$	angle of attack (deg)
$q(t)$	pitch rate (deg/sec)
$M(t)$	Mach number
$\delta_c(t)$	commanded tail deflection angle (deg)
$\delta(t)$	actual tail deflection angle (deg)
$\eta_c(t)$	commanded normal acceleration in g's
$\eta(t)$	actual normal acceleration in g's.

The variables $\eta(t)$ and $q(t)$ are measured, thus available for feedback use. Angle of attack $\alpha(t)$, Mach number $M(t)$ are variables to be used for scheduling purposes. The input to the plant is commanded tail deflection $\delta_c(t)$.

Further description of various constants, including their numerical values, is provided in Table 6.4.

The performance goals for the closed-loop system are:

$K_\alpha = (0.7)P_0S/mv_s$	
$K_q = (0.7)P_0Sd/I_y$	
$K_z = (0.7)P_0S/m$	
$A_x = (0.7)P_0SC_a/m$	
$P_0 = 973.3\ lbs/ft^2$	static pressure at $20,000ft$
$S = 0.44\ ft^2$	surface area
$m = 13.98\ slugs$	mass
$v_s = 1036.4\ ft/s$	speed of sound at $20,000ft$
$d = 0.75\ ft$	diameter
$I_y = 182.5\ slug.ft^2$	pitch moment of inertia
$C_a = -0.3$	drag coefficient
$\zeta = 0.7$	actuator damping ratio
$\omega_a = 150\ rad/s$	actuator undamped natural frequency
$a_n = 0.000103\ deg^{-3}$	
$b_n = -0.00945\ deg^{-2}$	
$c_n = -0.1696\ deg^{-1}$	
$d_n = -0.034\ deg^{-1}$	
$a_m = 0.000215\ deg^{-3}$	
$b_m = -0.0195\ deg^{-2}$	
$c_m = 0.051\ deg^{-1}$	
$d_m = -0.206\ deg^{-1}$	

Table 6.4: Coefficients of pitch-axis missile model

- Maintain robust stability over the operating range specified by $(\alpha(t), M(t))$ such that $-20° \leq \alpha(t) \leq 20°$ and $2 \leq M(t) \leq 4$. Robust stability is shown by varying angle of attack and tail-deflection component in coefficients C_m and C_n by $\pm 25\%$ and 10% independently.

- Track step commands in $\eta_c(t)$ with time constant no greater than $0.35\ sec$, maximum overshoot no greater than 10%, and steady-state error no greater than 1%.

- Maximum tail deflection rate for $1g$ step command in $\eta_c(t)$ does not exceed $25\ deg/sec$.

Let $\rho_1 := \alpha,\ \rho_2 := M$, then the LPV form of missile model (6.2.1)-(6.2.3) is written in the following equations.

$$\frac{d}{dt}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix} = \begin{bmatrix} K_\alpha\rho_2(t)\left[a_n\rho_1^2(t)+b_n|\rho_1(t)|+c_n\left(2-\frac{\rho_2(t)}{3}\right)\right]cos(\rho_1(t)) & 1\\ K_q\rho_2^2(t)\left[a_m\rho_1^2(t)+b_m|\rho_1(t)|+c_n\left(-7+\frac{8\rho_2(t)}{3}\right)\right] & 0\end{bmatrix}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}K_\alpha\rho_2(t)d_n cos(\rho_1(t))\\ K_q\rho_2^2(t)d_m\end{bmatrix}\delta(t), \tag{6.2.5}$$

$$\begin{bmatrix}\eta(t)\\ q(t)\end{bmatrix} = \begin{bmatrix} K_z\rho_2^2\left[a_n\rho_1^2(t)+b_n|\rho_1(t)|+c_n\left(2-\frac{\rho_2(t)}{3}\right)\right] & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix} + \begin{bmatrix}K_z\rho_2^2(t)d_n\\ 0\end{bmatrix}\delta(t). \tag{6.2.6}$$

where the parameter set $\mathcal{P} = [-20,\ 20]\times[2,\ 4]$. As the first parameter ρ_1 is actually one of the states in missile model, we call this plant "quasi-LPV" system.

Similar to the synthesis procedure for LTI systems, we use rational weighting functions to characterize the overall closed-loop performance objective. The open-loop interconnection for synthesis is shown in Figure 6.2

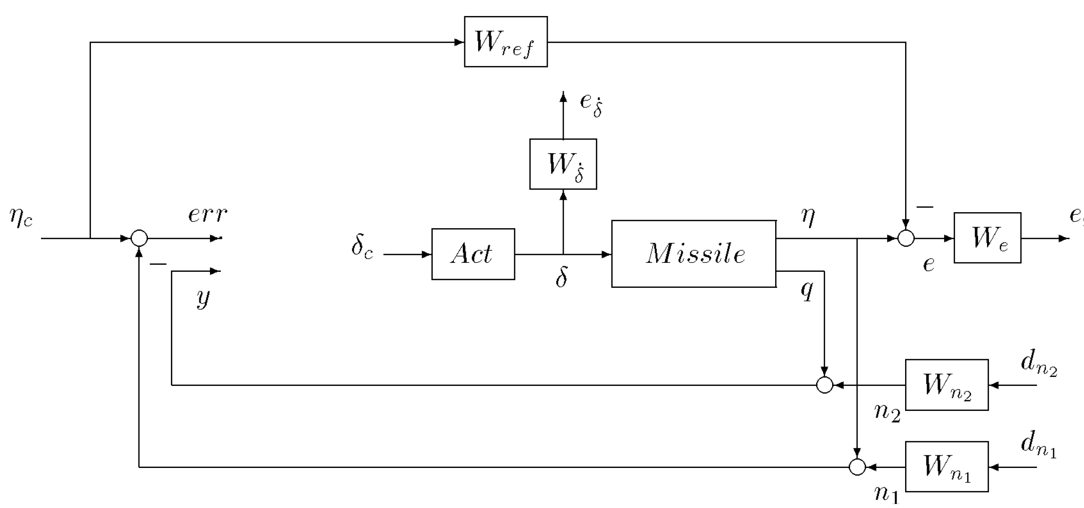

Figure 6.2: Weighted open-loop interconnection of missile plant.

where

$$W_{ref}(s) := \frac{144(-0.05s+1)}{s^2+2\times0.8\times12s+144},$$

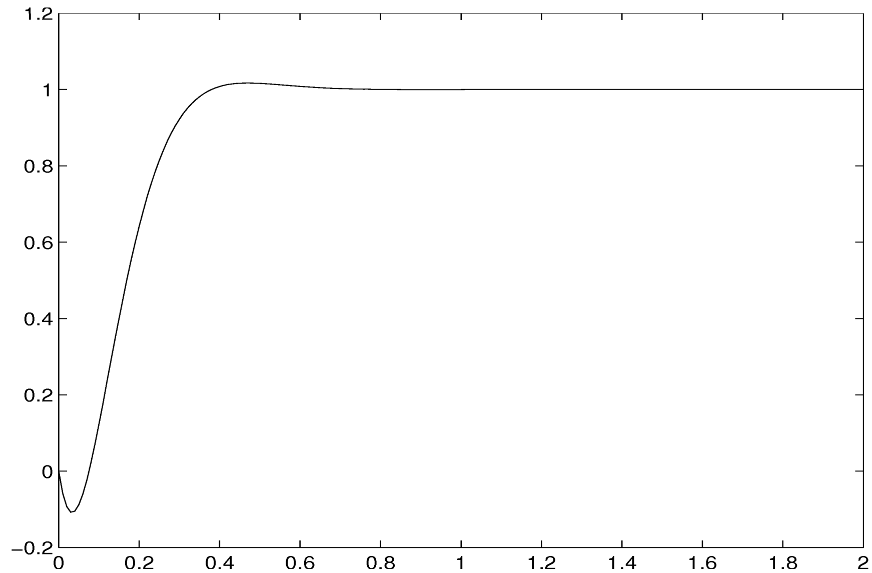

Figure 6.3: Unit step response through weighting function W_{ref}.

$$\begin{aligned} W_e(s) &:= \frac{0.5s + 17.321}{s + 0.0577}, \\ W_{\dot{\delta}}(s) &:= \frac{s}{25(0.005s + 1)}, \\ W_{n_1}(s) &= W_{n_2}(s) := 0.001, \\ Act(s) &:= 1. \end{aligned}$$

Specifically, the weighting function W_{ref} reflects the designated step response (Figure 6.3), which exhibits no more than $0.35sec$ time constant and non-minimum phase characteristics of the missile plant. The non-minimum phase property can be verified by local Jacobian linearization of missile model, in which the right-half plane zero ranges from 20.0 to 46.0. With no real theory to guide us, we simply pick the slowest zero ($s = 20.0$) in the desired command response filter. The weighting function W_e has a low frequency gain 300, which corresponds to tighter, 0.33%, tracking error, and high frequency gain 0.5 to limit overshoot less than 5%. In order to keep the order of weighted system as low as possible, we deliberately exclude actuator dynamics in the open-loop interconnection. The actuator model (6.2.4) will only be used for simulation purpose.

The weighted open-loop LPV system has 6 states in which 2 states are from missile plant, and remaining 4 states from weighting functions. The system has 6 inputs and 6 outputs. The LPV controller measures 2 outputs (normal acceleration tracking error err and pitch rate measurement y) and generates one control input (commanded tail deflection δ_c).

6.2.2 Synthesis and Simulation Results

To do the synthesis for the weighted LPV system, which has nonlinear dependence on parameters α, M, we need to grid the two-dimensional parameter space $\mathcal{P}$. It is easy to observe that the state-space data of missile model in LPV form (6.2.5)-(6.2.6) are symmetric with respect to ρ_1, so we only have to consider half parameter space. Particularly, gridding the half of parameter space $\mathcal{P}$ by 6×6 points, that is

$$\mathcal{V} := \{(\rho_1, \rho_2) : \rho_1 \in \{0, 4, 8, 12, 16, 20\}, \rho_2 \in \{2, 2.4, 2.8, 3.2, 3.6, 4\}\}.$$

This is actually very rough gridding for the parameter space, so we need to verify the resulting solutions X and Y on finer griddings after the synthesis.

Implementing Theorem 5.3.7 in [Bec] with LMIlab [GahNLC], we solve X, Y which satisfy the constraints on all $\rho \in \mathcal{V}$ and optimal $\gamma = 3.13$ within 0.1% error. Furthermore, we check resulted X, Y and γ value over 100×100 uniformly spaced points of $\mathcal{P}$. The maximum eigenvalue of state-feedback, output estimation and negative coupling conditions ranges from -5.57×10^{-7} to -6.17×10^{-2}, which indicates that the LMIs are indeed solved over the whole parameter space $\mathcal{P}$. So the closed-loop LPV system's induced $\mathcal{L}_2$-norm, from disturbance to error, is guaranteed less than 3.13 for arbitrarily fast-varying parameter trajectory in the set $\mathcal{P}$.

An admissible parameter-dependent controller can be constructed using resulted X, Y [Bec, Theorem 5.3.7]. Then we would like to analyze its property from several aspects.

Induced $\mathbf{L_2}$-Norm Performance at Frozen Parameters

For each fixed point in $\mathcal{P}$, the system is in LTI form. So we can synthesize optimal $\mathcal{H}_\infty$ controller for this point. Also by evaluating LPV controller at this parameter, we come up with a (sub) optimal controller for resulted LTI system. The closed-loop $\mathcal{H}_\infty$-norm at 9 selected points are shown in Tables 6.5 and 6.6.

The optimal γ performance for all fixed points ranges from 0.30 to 0.96. The performance level using LPV controller at fixed parameter values is between 0.89 to 2.66, which is less than 3.13 as expected. The performance of LPV controller is 98% to 414%

		Mach		
		2.0	3.0	4.0
	0.0	0.9570	0.4492	0.3320
α	10.0	0.7617	0.3320	0.3027
	20.0	0.6641	0.3027	0.3027

Table 6.5: Frozen optimal closed-loop $\mathcal{H}_\infty$-norm.

		Mach		
		2.0	3.0	4.0
	0.0	2.6613	0.8898	1.1704
α	10.0	2.0870	0.8899	1.4335
	20.0	1.7809	0.8900	1.5574

Table 6.6: Frozen LPV closed-loop $\mathcal{H}_\infty$-norm.

higher than optimal one. So the LPV controller is not optimal for fixed parameters, but remember that LPV controller is designed for time-varying parameters.

Controller Frequency Response

The controller has two inputs (acceleration tracking error err and measurement y of pitch rate) and one output (tail-deflection command δ_c). For 9 fixed parameter values, we get optimal controller and LPV controller evaluated at these points. The frequency responses of both controllers from err to δ_c are shown in Figures 6.24, 6.25 6.26, 6.27. The frequency responses from y to δ_c are shown in Figures 6.28, 6.29, 6.30, 6.31.

Note that the frequency responses of both controllers are quite different. The transfer functions of the second channel in the optimal controller are zeros for some fixed parameters, which means the missile is solely controlled by the first channel signal at these points.

Step Response

Next, we will compare the closed-loop response for unit step acceleration command using both optimal and LPV controllers. Recall the performance objective is: no more than $0.35sec$ time constant, less than 10% over-shoot and no more than 1% steady-state tracking error, tail-deflection rate should be less than 25 $deg/sec/g$. The tracking behavior $\eta(t)$ of both controllers under $1g$ step input at 9 fixed points are given in Figures 6.32, 6.33. The corresponding tail-deflection rates are shown in Figures 6.34 and 6.35.

From the time responses, we observe that the majority of the performance objectives are met although LPV controller has certain degree of performance degradation (slower rise time, higher command signal) compared with optimal controller,

Now, we are in the position to do non-linear simulations for the missile problem. While Mach number is an exogenous scheduling variable in the design, for simulation purposes we set it to be:

$$\begin{aligned} \dot{M}(t) &= \frac{1}{v_s}\left[-|\eta(t)|sin(|\alpha(t)|) + A_x M^2(t)\ cos(\alpha(t))\right] \\ M(0) &= M_0, \end{aligned}$$

this will provide a reasonably realistic Mach profile [NicRR].

Recall in LPV control theory, the parameters are assumed to be measurable in real-time. But for the missile problem we are looking at, one of LPV plant parameters, namely, angle of attack α is not measurable. In output equation (6.2.3),

$$\eta = f(\alpha, \delta, M) = K_z M^2 C_n(\alpha, \delta, M),$$

all variables are measurable except α. By inverse function theorem, α is solvable in terms of η, δ and M. The polynomial approximation of the inverse function $\alpha = f^{-1}(\eta, \delta, M)$ is given by

$$\begin{aligned} \alpha_e &= -1.396 - 0.33421M_N - 3.7653\delta_N - 0.91681\delta_N M_N \\ &+ \eta_N\left(-46.03 + 21.26M_N - 8.8362M_N^2 - 0.33564\delta_N + 0.385\delta_N M_N + 0.32892\delta_N M_N^2\right) \\ &+ \eta_N^3\left(61.367 - 69.756M_N + 30.44M_N^2 + 3.9589\delta_N - 15.668\delta_N M_N + 11.498\delta_N M_N^2\right) \\ &+ \eta_N^5\left(-54.655 + 94.381M_N - 48.212M_N^2 - 4.7973\delta_N + 18.807\delta_N M_N - 13.871\delta_N M_N^2\right). \end{aligned} \tag{6.2.7}$$

where the normalized variables $\eta_N := \eta/60.0$, $\delta_N := (\delta - 10.0)/25.0$ and $M_N := M - 3.0$. The relative approximation error ranges from 0.002% to 38%, and the mean of errors is about 8%. So the curve fitting is ok but not perfect. The α estimator (6.2.7) is kept the same throughout simulations even in the cases where we perturb the missile aerodynamic coefficients C_n and C_m.

In Figure 6.36, we plot the commanded acceleration $\eta_c(t)$ (dash-dot line) and missile's actual acceleration $\eta(t)$ using LPV controller. Figures 6.37, 6.38 show the corresponding angle of attack $\alpha(t)$, and tail-deflection rate $\dot{\delta}(t)$ with given acceleration command.

From the simulation results, It is clear that tracking performance and tail-deflection rate requirements are satisfied very well.

The robustness property of the LPV controller is shown by perturbing aerodynamic coefficients C_m and C_n independently. For C_m, we simultaneously perturb a_m, b_m and c_m from their nominal values by a factor of 1.25 and 0.75, d_m by a factor of 1.25 and 0.75 from its nominal value. Similar perturbations to C_n are carried out, but the variations are limited to $\pm 10\%$. Totally, 16 plots result from the combination of all of these variations, which are shown in Figures 6.39, 6.40, 6.41.

6.3 Benchmark For Comparison

We have developed many types of method to design controller for LPV systems with induced $\mathbf{L}_2$-norm performance. It is useful to compare all of them at this stage. For such a purpose, We use the two-disk problem given in 6.1 as our benchmark. The plant model is given in equation (6.1.3) and the open-loop interconnection is in Figure 6.1 with weighting functions as

$$\begin{aligned}
W_e(s) &:= \frac{0.3s + 1.2}{s + 0.04}, \\
W_u(s) &:= \frac{s + 0.1}{0.01s + 125}, \\
W_a(s) &:= 0.00001, \\
W_n(s) &:= \frac{s + 0.4}{0.01s + 400}, \\
Act(s) &:= \frac{1}{0.01s + 1}.
\end{aligned}$$

Using μ type synthesis method [WuP3], [BalDGPS], we can design a robust controller to tolerate the time-varying parameters in given bounded set $\mathcal{P}$. By measuring parameters in real-time, we may construct a parameter-dependent controller to gain-schedule the control action for LPV systems. The performance of LPV control can further be improved with real-time information of parameter derivatives. If the parameter is fixed, then $\mathcal{H}_\infty$ optimal control design for LTI systems is applicable.

For this problem, we pick $N = 3$ and choose the basis functions as

$$f_1(\rho) = g_1(\rho) := 1, \quad f_2(\rho) = g_2(\rho) := \rho_1, \quad f_3(\rho) = g_3(\rho) := \rho_2.$$

These seem natural, given the parameter dependence of the plant on ρ. So the parameter-dependent functions $X(\rho)$ and $Y(\rho)$ are inthe form of

$$X(\rho) = \sum_{i=1}^{3} f(\rho) X_i, \qquad Y(\rho) = \sum_{i=1}^{3} g_i Y_i.$$

For synthesis, we use a rather coarse gridding the parameter space $\mathcal{P}$, namely 25 points with 5 points in each dimension uniformly. Using Theorem 4.3.2, we solve the Parameter-Dependent γ-Performance Problem at various variation rate levels ν_1, ν_2. The relationship among the optimal achievable performance γ and ν_1, ν_2 are shown in Table 6.7. From the

		ν_2		
		10^{-3}	3.1623	10^4
	10^{-3}	0.8853	0.8916	0.9932
ν_1	3.1623	0.8944	0.9023	1.0105
	10^4	0.9438	0.9525	1.1305

Table 6.7: Induced $\mathbf{L}_2$-norm performance with various parameter variation rates.

table, we clearly see the decrease of induced $\mathbf{L}_2$-norm as the bounds of variation rate being reduced.

The following Table 6.8 shows achievable induced-$\mathbf{L}_2$-norm using different control design techniques. Note the induced $\mathbf{L}_2$-norms keep decreasing as we have more and more information about parameters. Also we observe that exploiting the realness of parameter by LPV control and SQLF does not help much compared with LFT control method.

Last, we will show the performance of different control methods through nonlinear simulation. The reference command is a unit step input, and the parameter trajectories are

$$\rho_1(t) = 4.5\left(\sin(0.6t) + 1\right), \qquad \rho_2(t) = 12.5\left(\cos(0.2t) + 1\right).$$

Note that the derivative of parameters are less than 3.16, so the induced $\mathbf{L}_2$-norm using LPV controller with PDLF is guaranteed to be less than 0.9 from Table 6.7. The performance of quadratic LPV control is less than 1.13, while that of robust control less than 1.55. The performance of LTV control for this particular trajectory is 0.73.

Figure 6.42 shows the tracking performance of four controllers, that is, robust control, quadratic LPV control, LPV control with PDLF, and LTV control. Figure 6.43 is the corresponding control forces of four controllers.

	Induced $\mathbf{L}_2$-Norm	Parameter variation rates	
		Bound ν_1	Bound ν_2
Robust Control with constant D-scalings	1.55	∞	∞
LFT control	1.14	∞	∞
LPV control with SQLF	1.13	∞	∞
LPV control with PDLF	$0.89 - 1.13$	10^4	10^4
LTV control for a particular trajectory	0.73	2.7	2.5
Optimal $\mathcal{H}_\infty$ control for fixed parameters	$0.74 - 0.82$	0	0

Table 6.8: Comparison of performance of different control methods

From the simulation results, we observe that the LTV control is the best among all control configurations, though it is only suitable for this particular parameter trajectory and must be calculated in advance. LPV controller with PDLF performs better than quadratic LPV control, because of the use of parameter dependent Lyapunov function to exploit bounded parameter variation rates information. The performance of Robust control is the worst but requires least information of parameters.

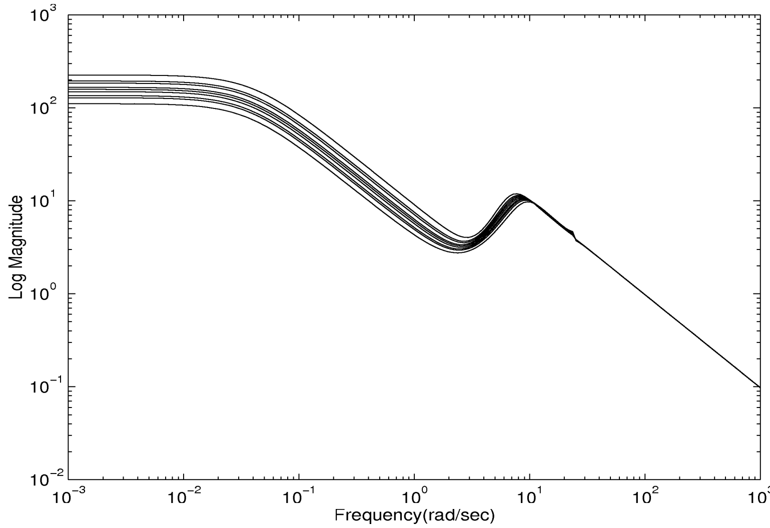

Figure 6.4: $\mathcal{H}_2$ optimal controller: magnitude plot from reference command ref to control force u_c at 9 fixed parameter values.

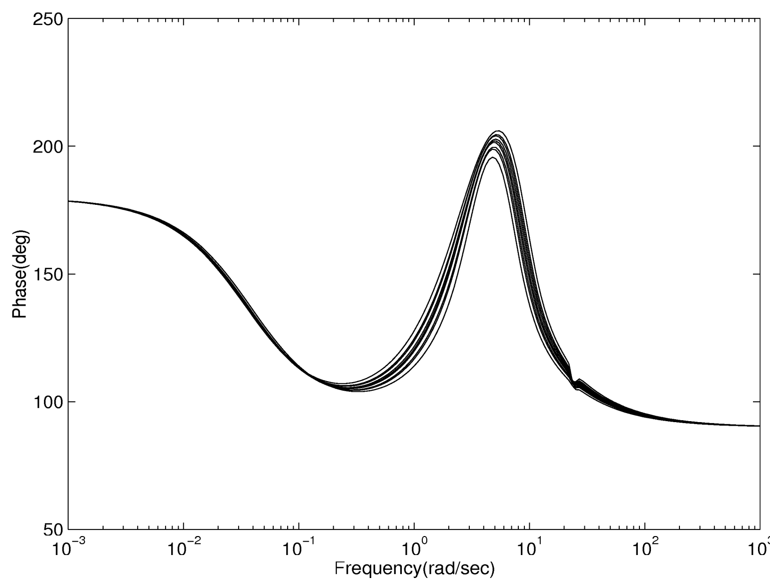

Figure 6.5: $\mathcal{H}_2$ optimal controller: phase plot from reference command ref to control force u_c at 9 fixed parameter values.

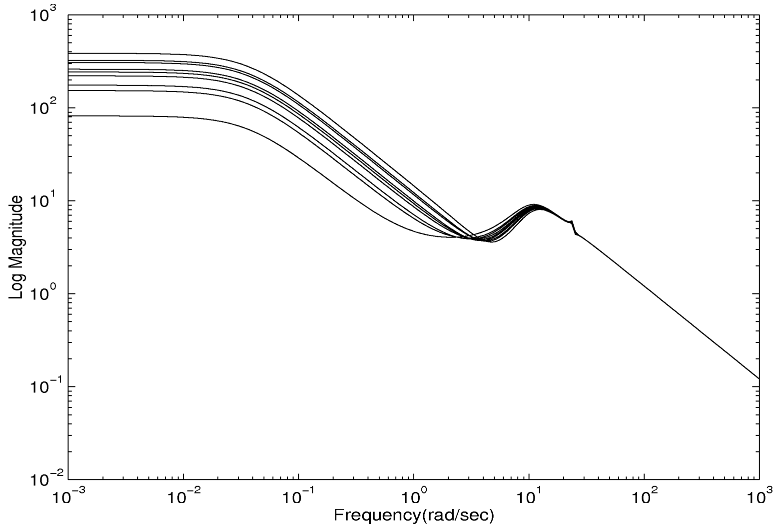

Figure 6.6: Output-feedback LPV controller: magnitude plot from reference command ref to control force u_c at 9 fixed parameter values.

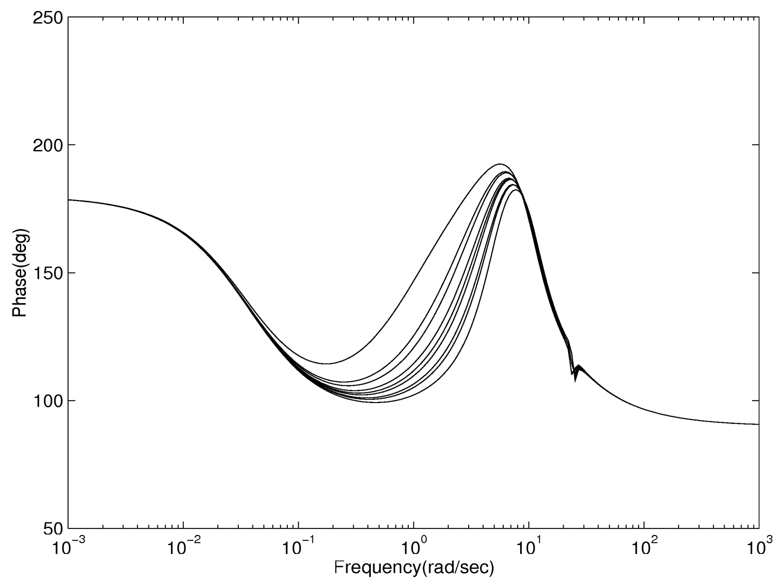

Figure 6.7: Output-feedback LPV controller: phase plot from reference command ref to control force u_c at 9 fixed parameter values.

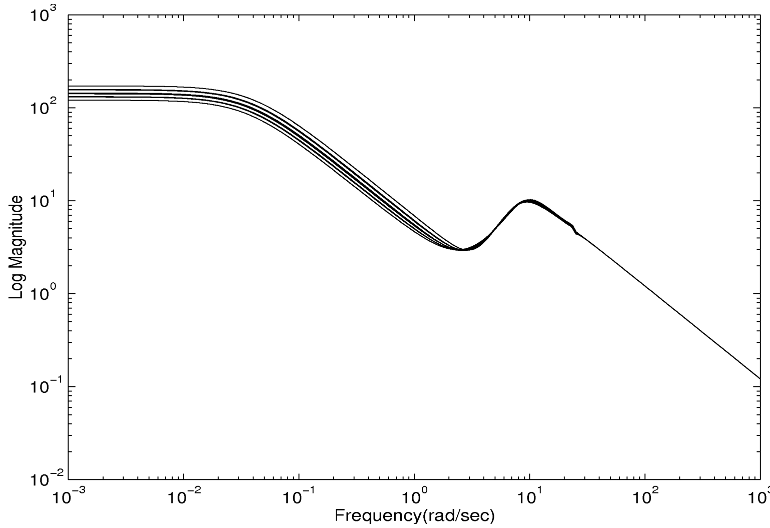

Figure 6.8: State-feedback LPV controller plus Kalman filter: magnitude plot from reference command ref to control force u_c at 9 fixed parameter values.

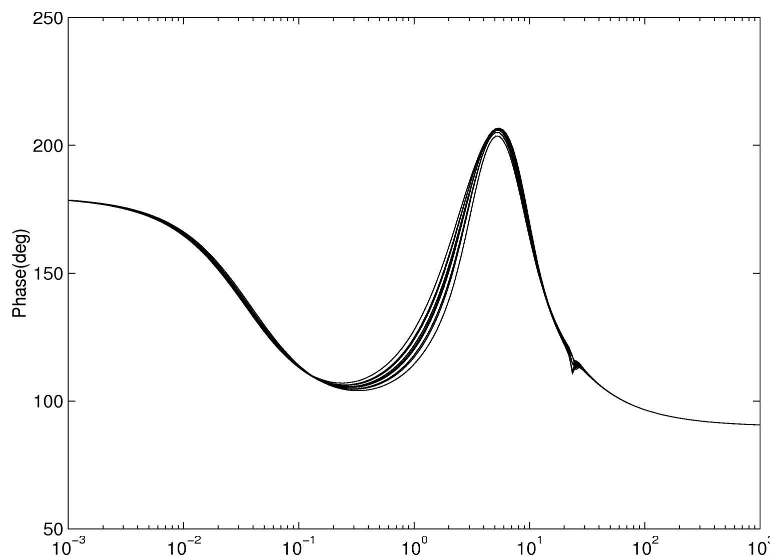

Figure 6.9: State-feedback LPV controller plus Kalman filter: phase plot from reference command ref to control force u_c at 9 fixed parameter values.

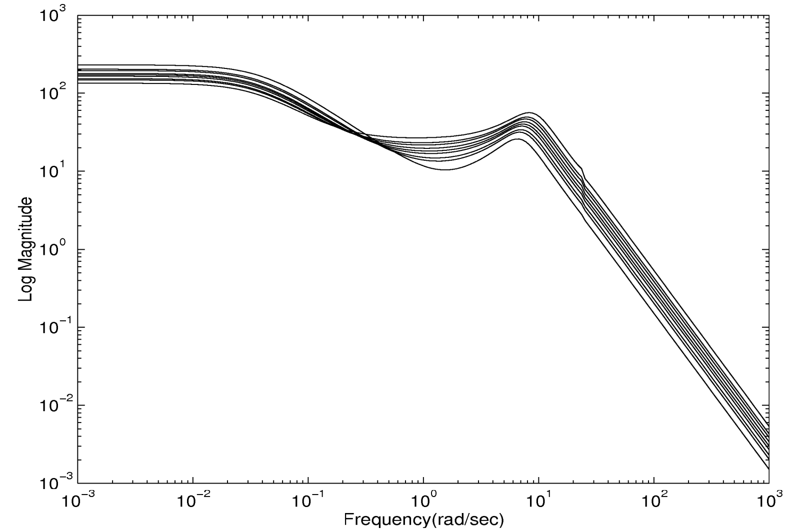

Figure 6.10: $\mathcal{H}_2$ optimal controller: magnitude plot from measurement y to control force u_c at 9 fixed parameter values.

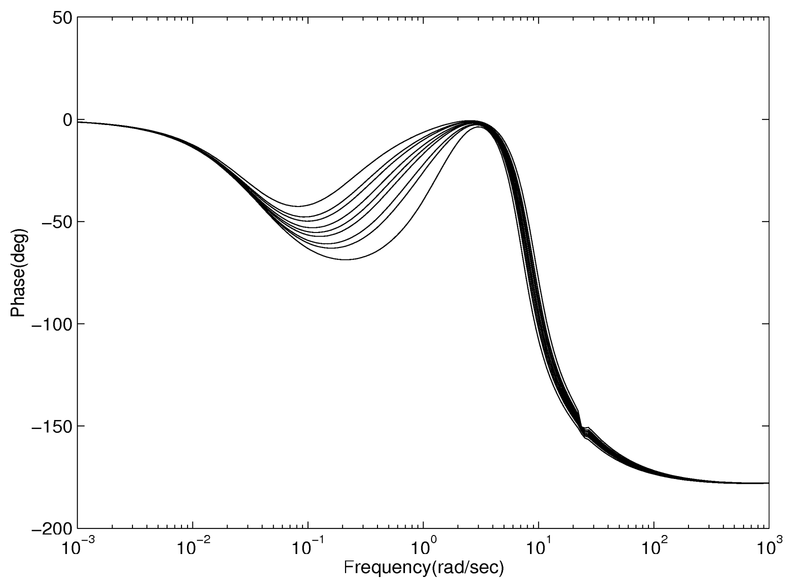

Figure 6.11: $\mathcal{H}_2$ optimal controller: phase plot from measurement y to control force u_c at 9 fixed parameter values.

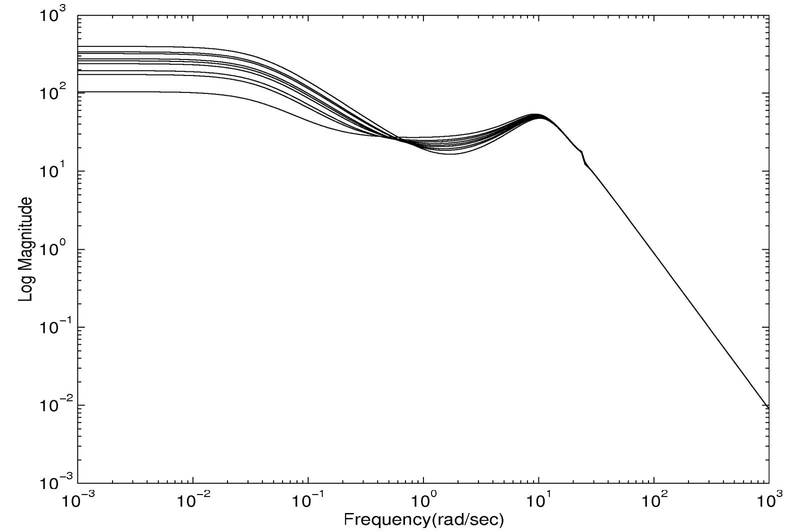

Figure 6.12: Output-feedback LPV controller: magnitude plot from measurement y to control force u_c at 9 fixed parameter values.

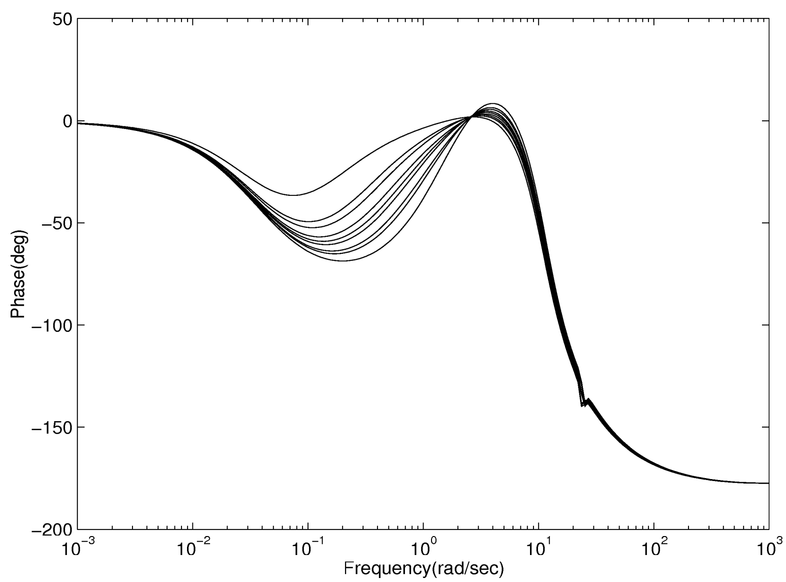

Figure 6.13: Output-feedback LPV controller: phase plot from measurement y to control force u_c at 9 fixed parameter values.

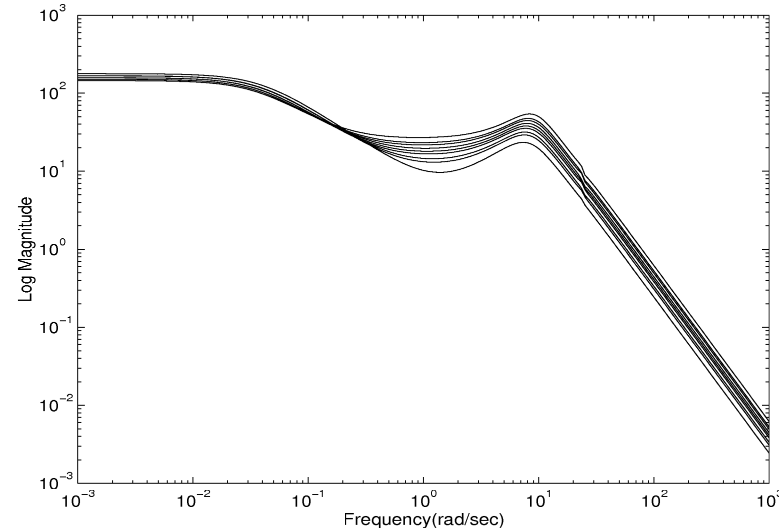

Figure 6.14: State-feedback LPV controller plus Kalman filter: magnitude plot from measurement y to control force u_c at 9 fixed parameter values.

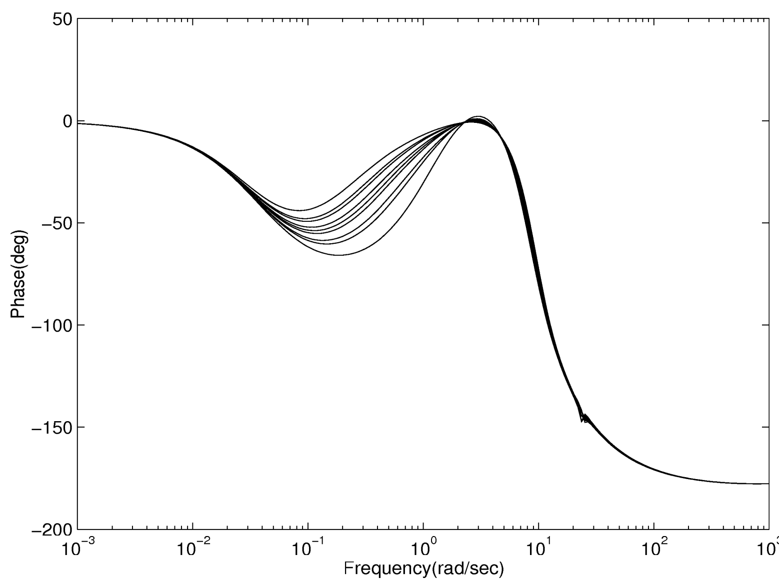

Figure 6.15: State-feedback LPV controller plus Kalman filter: phase plot from measurement y to control force u_c at 9 fixed parameter values.

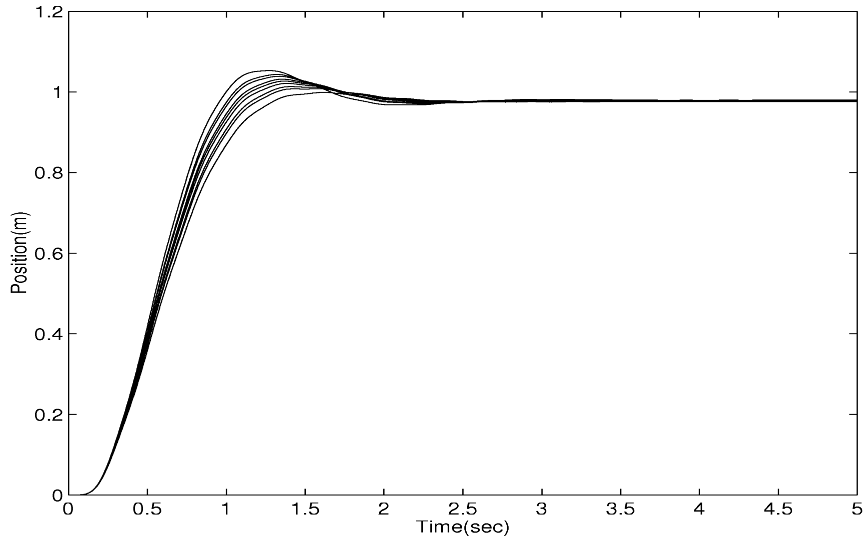

Figure 6.16: Fixed parameter step response $r_2(t)$ using $\mathcal{H}_2$ optimal controller.

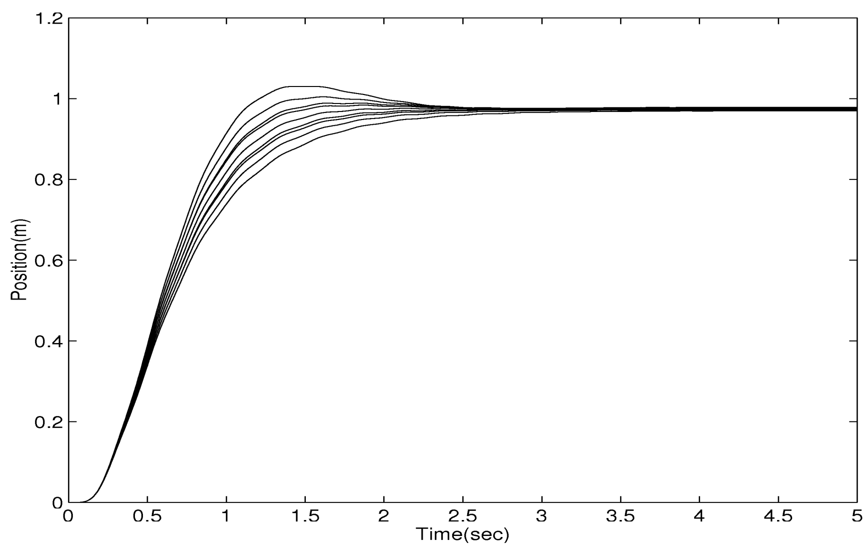

Figure 6.17: Fixed parameter step response $r_2(t)$ using output-feedback LPV controller.

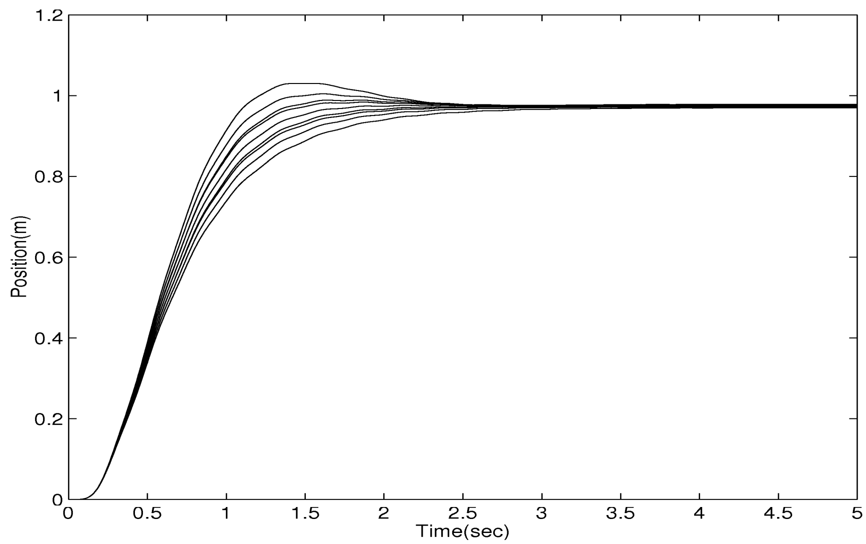

Figure 6.18: Fixed parameter step response $r_2(t)$ using state-feedback LPV controller plus Kalman filter.

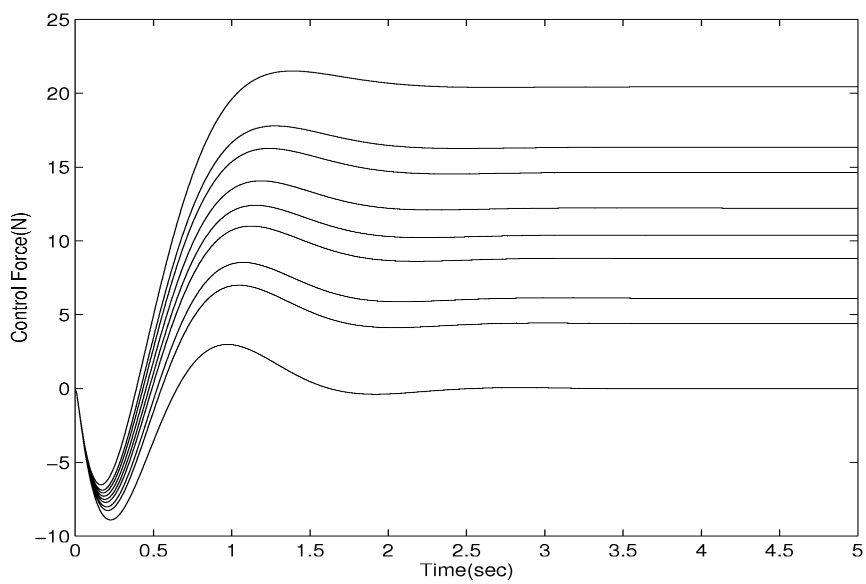

Figure 6.19: Fixed parameter control force $u(t)$ using $\mathcal{H}_2$ optimal controller.

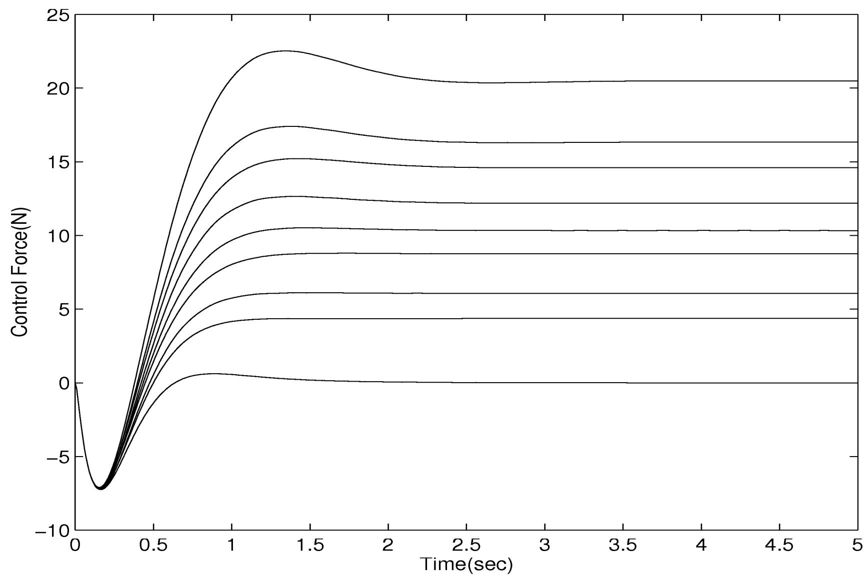

Figure 6.20: Fixed parameter control force $u(t)$ using output-feedback LPV controller.

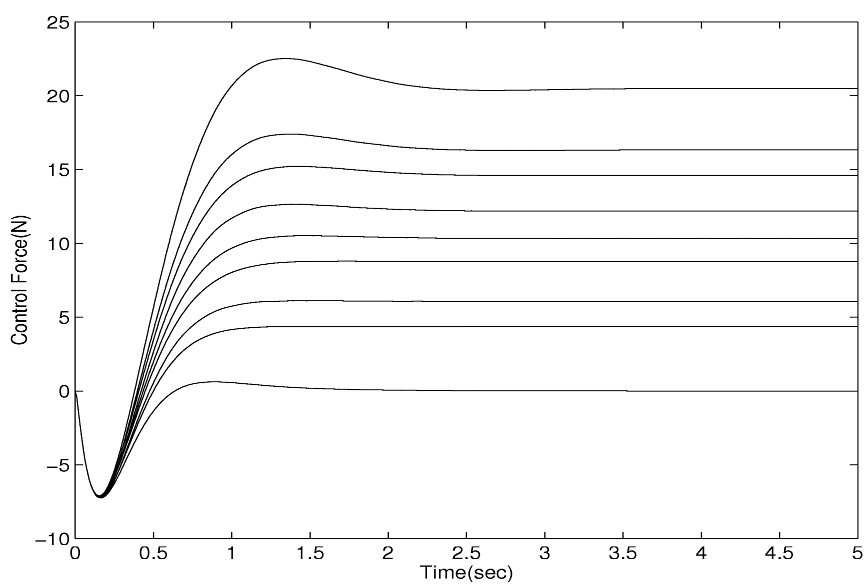

Figure 6.21: Fixed parameter control force $u(t)$ using state-feedback LPV controller plus Kalman filter.

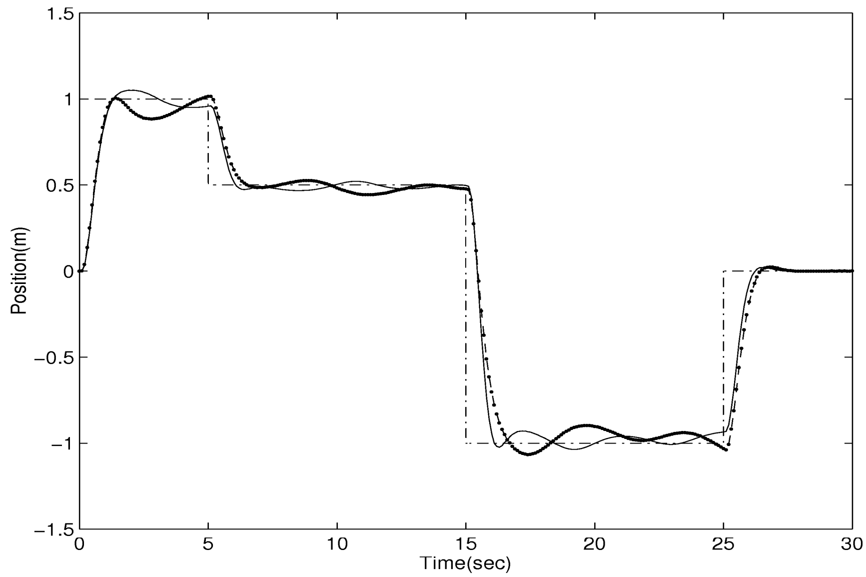

Figure 6.22: Position $r_2(t)$ using output-feedback LPV controller (dash line), state-feedback LPV control with Kalman filter (dot line) and optimal LTV controller (solid line) to track a given command.

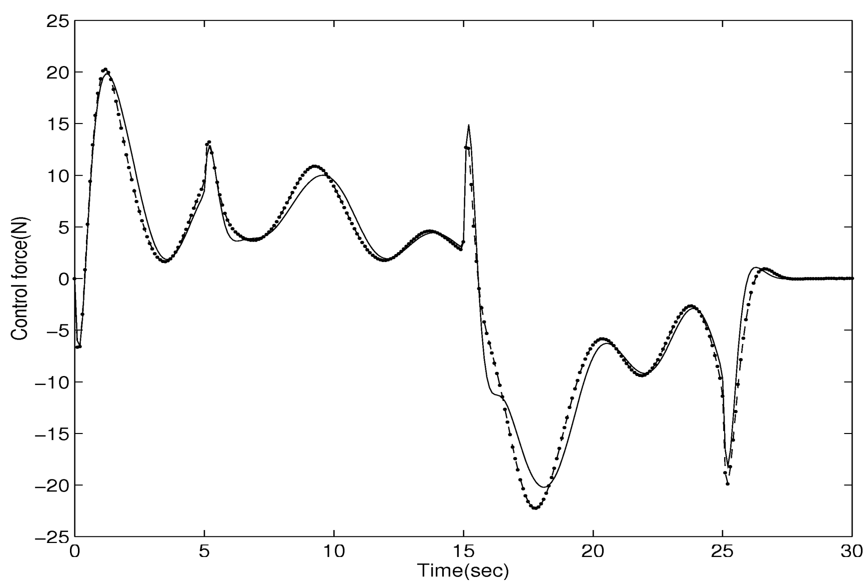

Figure 6.23: Actual control force $u(t)$ of output-feedback LPV controller (dash line), state-feedback LPV control with Kalman filter (dot line) and optimal LTV controller (solid line) for a given command.

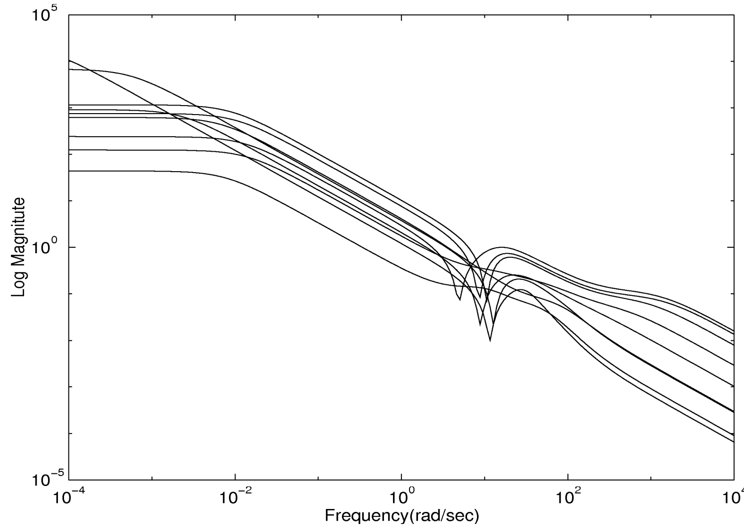

Figure 6.24: $\mathcal{H}_\infty$ optimal controller: magnitude plot from tracking error err to acceleration command δ_c at fixed parameter values.

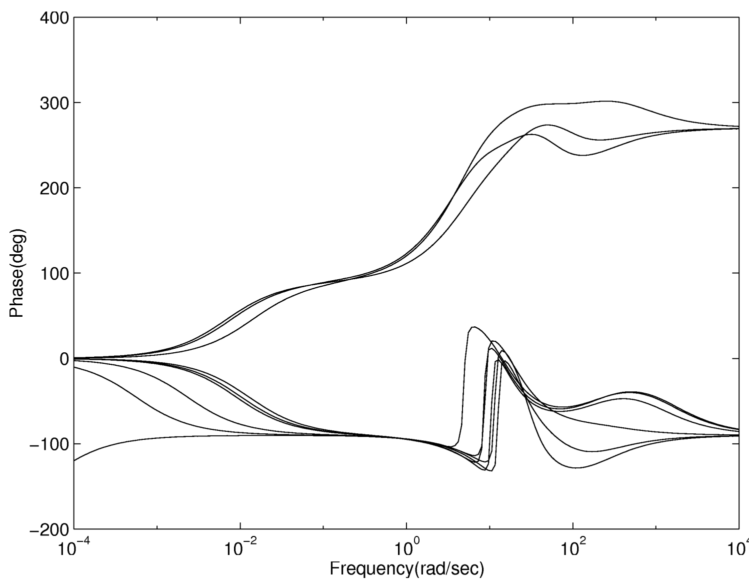

Figure 6.25: $\mathcal{H}_\infty$ optimal controller: phase plot from tracking error err to tail-deflection command δ_c at 9 fixed parameter values.

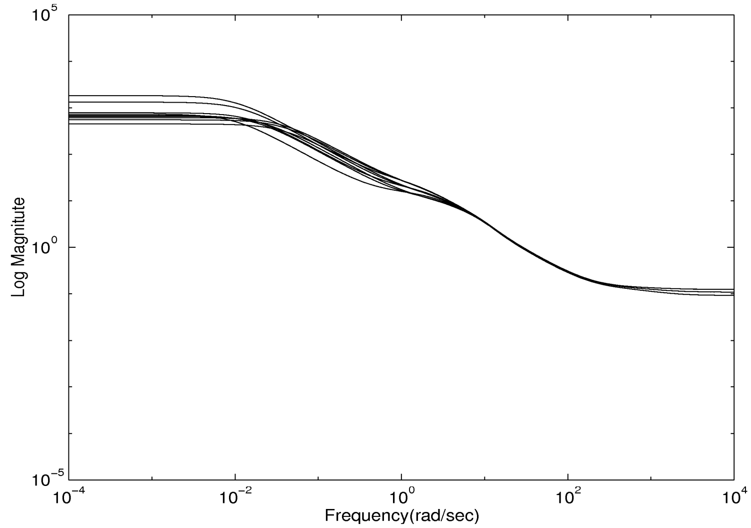

Figure 6.26: LPV controller: magnitude plot from tracking error err to tail-deflection command δ_c at 9 fixed parameter values.

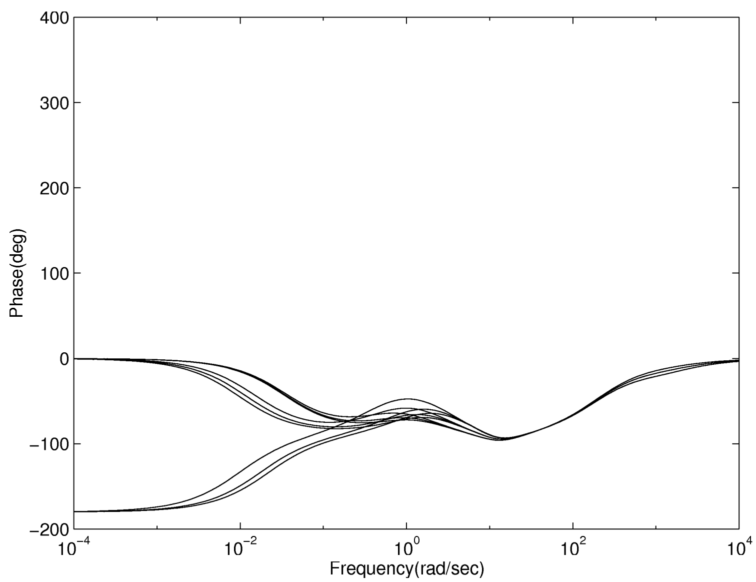

Figure 6.27: LPV controller: phase plot from tracking error err to tail-deflection command δ_c at 9 fixed parameter values.

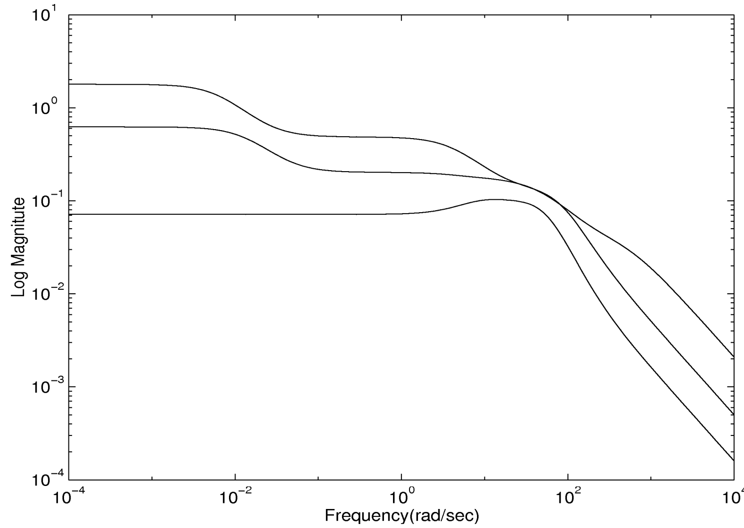

Figure 6.28: $\mathcal{H}_\infty$ optimal controller: magnitude plot from measurement y to tail-deflection command δ_c at 9 fixed parameter values.

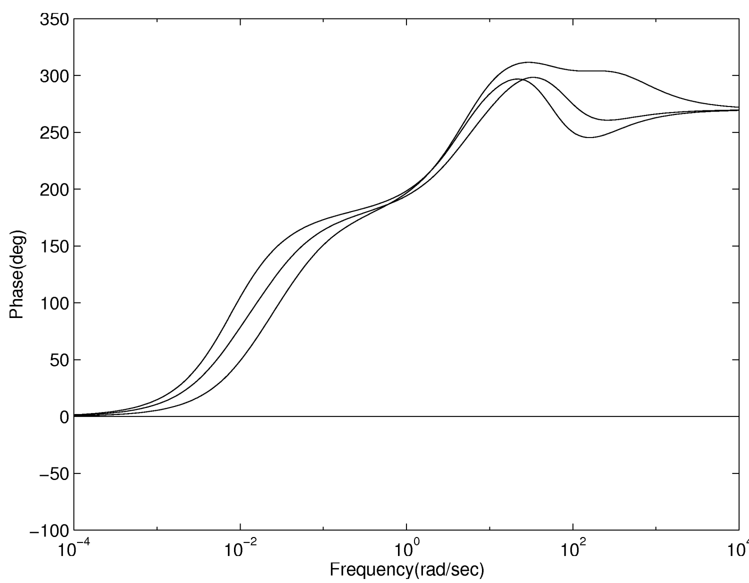

Figure 6.29: $\mathcal{H}_\infty$ optimal controller: phase plot from measurement y to tail-deflection command δ_c at 9 fixed parameter values.

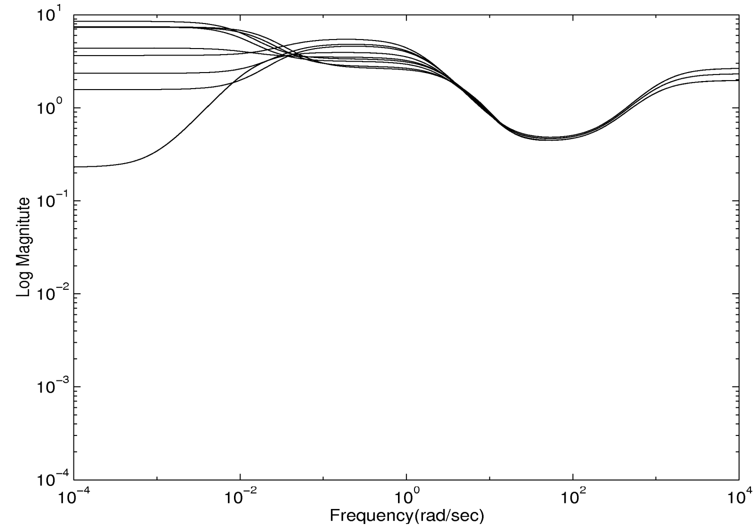

Figure 6.30: LPV controller: magnitude plot from measurement y to tail-deflection command δ_c at 9 fixed parameter values.

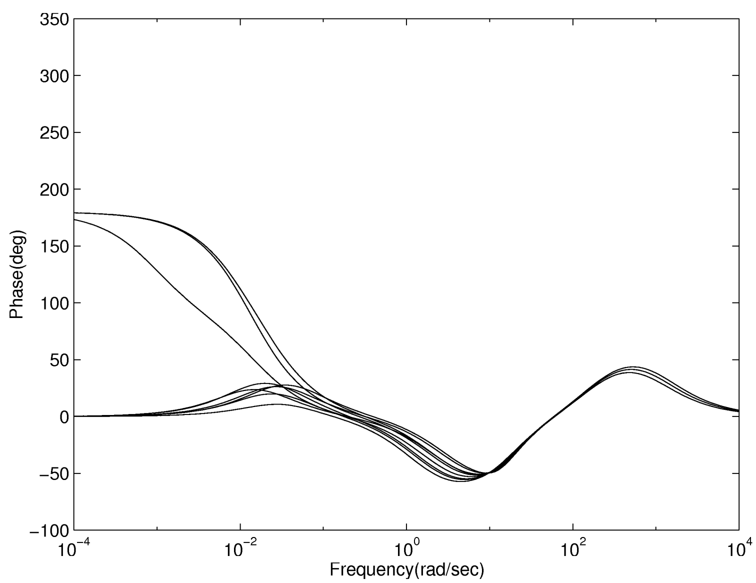

Figure 6.31: LPV controller: phase plot from measurement y to tail-deflection command δ_c at 9 fixed parameter values.

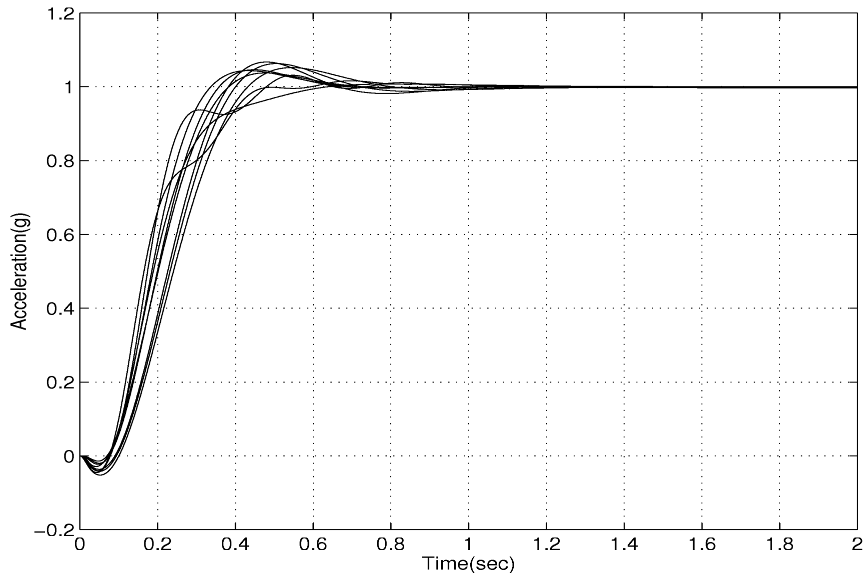

Figure 6.32: Fixed parameter $1g$ step response $\eta(t)$ for $\mathcal{H}_\infty$ optimal controller.

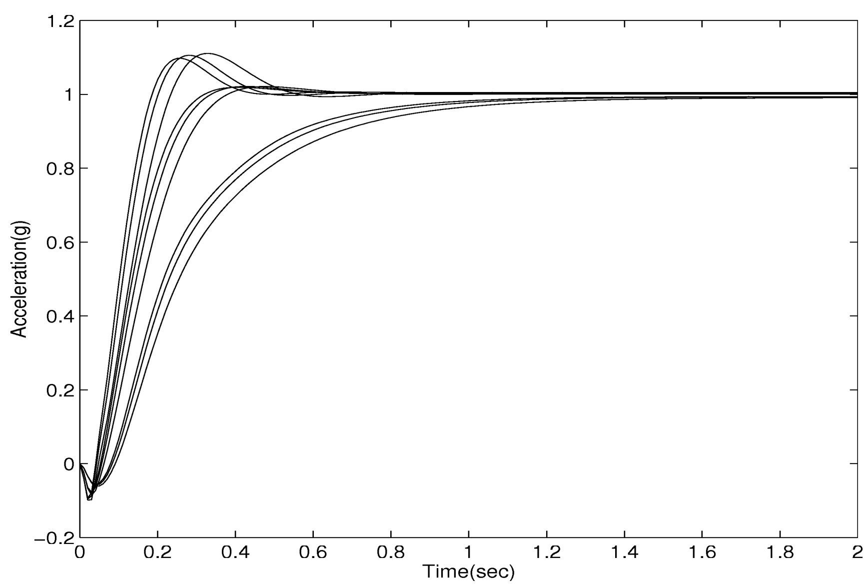

Figure 6.33: Fixed parameter $1g$ step response $\eta(t)$ for LPV controller.

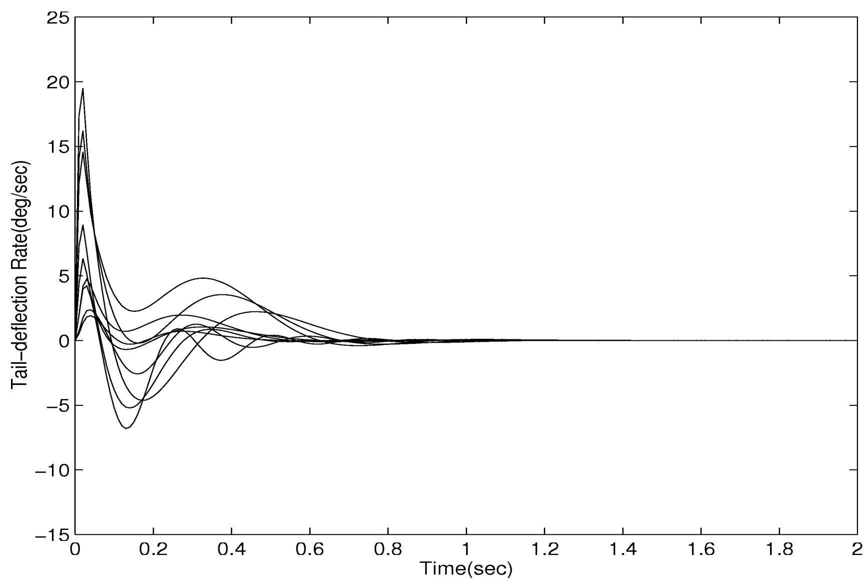

Figure 6.34: Fixed parameter tail-deflection rate $\dot{\delta}(t)$ for $\mathcal{H}_\infty$ optimal controller.

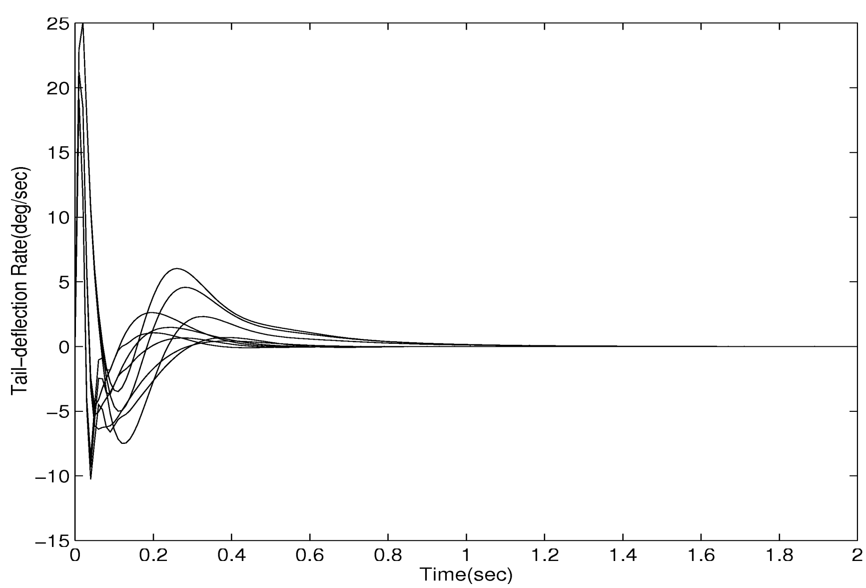

Figure 6.35: Fixed parameter tail-deflection rate $\dot{\delta}(t)$ for LPV controller.

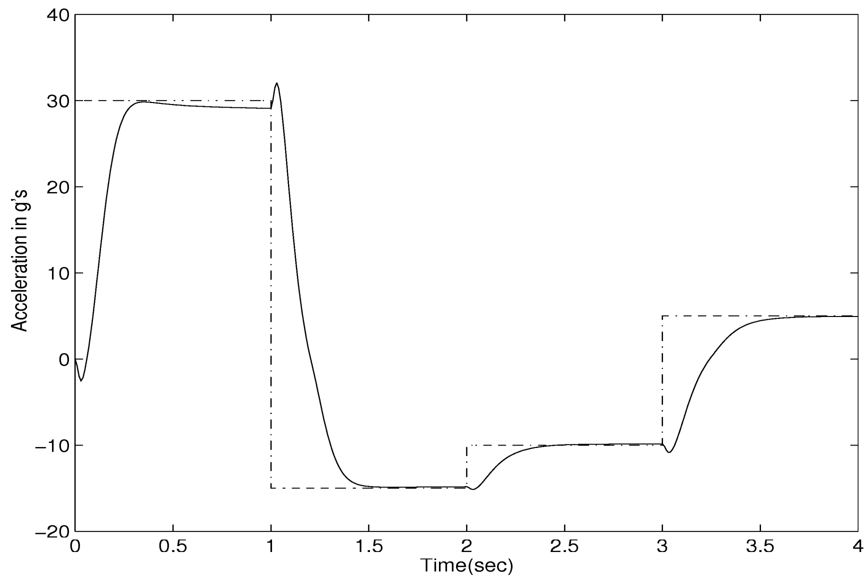

Figure 6.36: Normal acceleration $\eta(t)$ tracking a sequence of step acceleration commands $\eta_c(t)$.

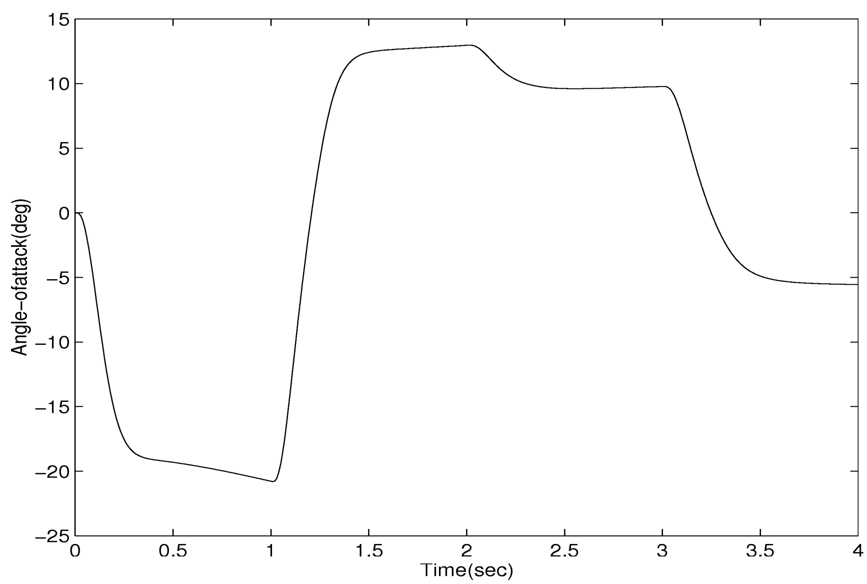

Figure 6.37: Angle-of-attack $\alpha(t)$ with respect to a given acceleration command.

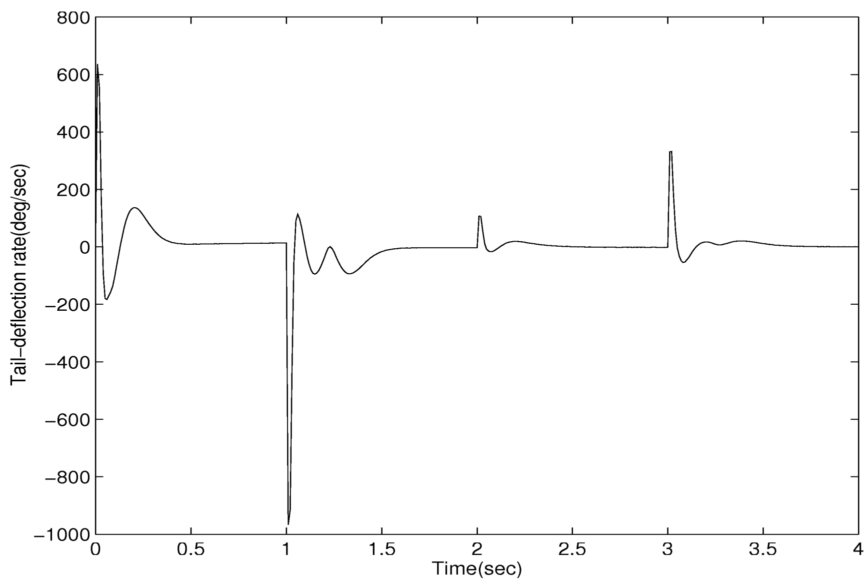

Figure 6.38: Tail-deflection rate $\dot{\delta}(t)$ with respect to a given acceleration command.

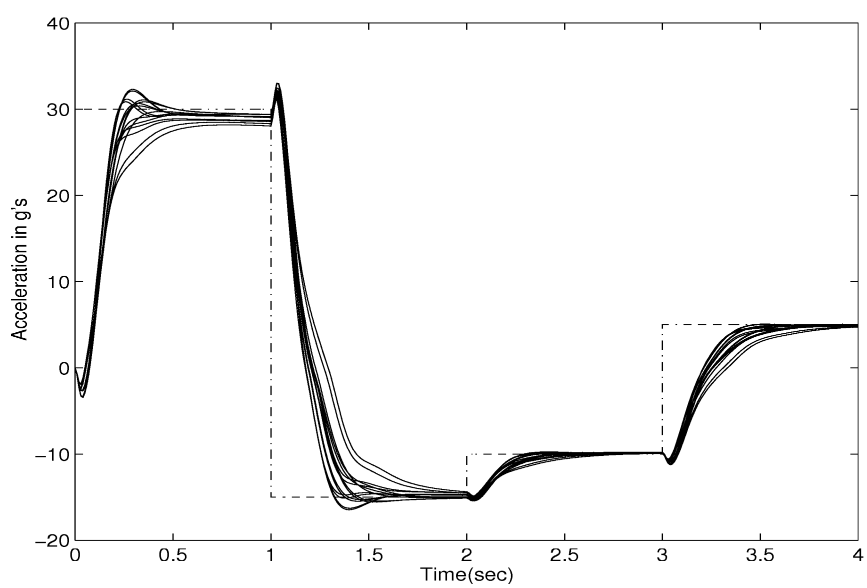

Figure 6.39: Normal acceleration $\eta(t)$ with perturbed C_n and C_m.

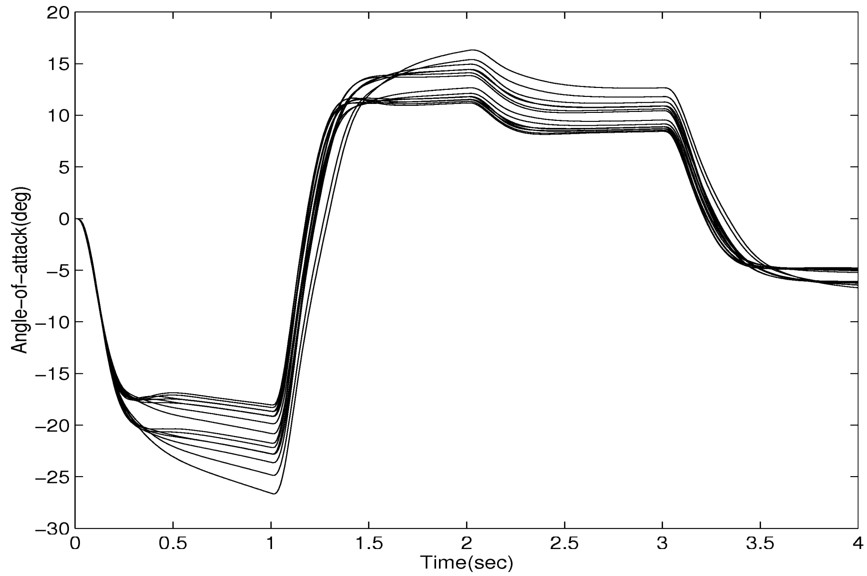

Figure 6.40: Angle-of-attack $\alpha(t)$ with perturbed C_n and C_m.

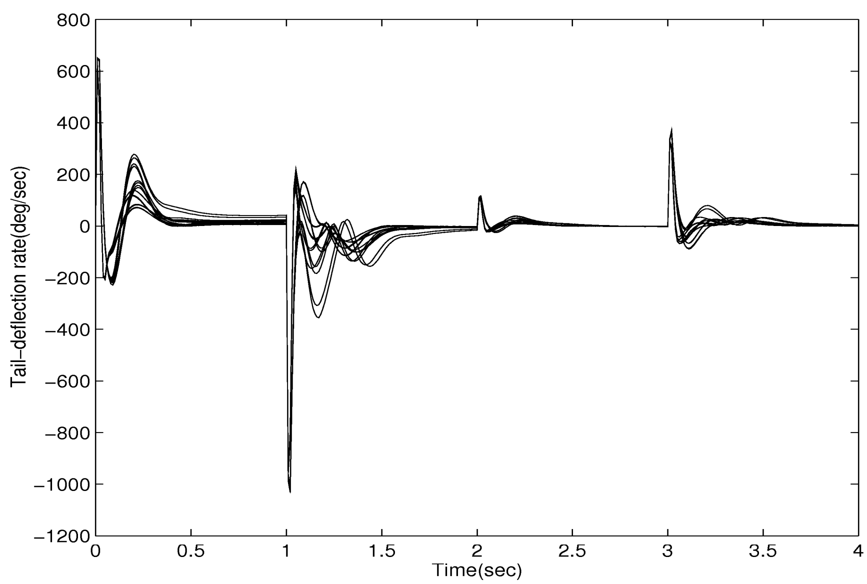

Figure 6.41: Tail-deflection rate $\dot{\delta}(t)$ with perturbed C_n and C_m.

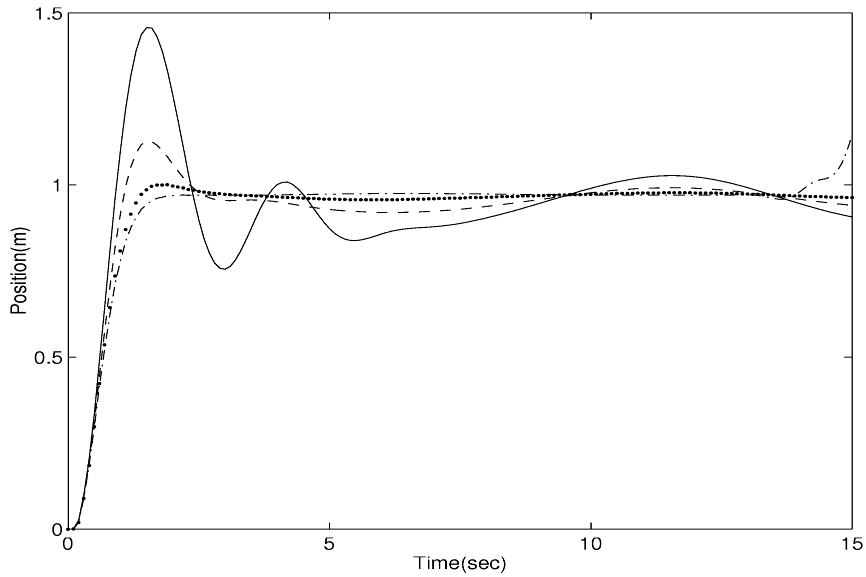

Figure 6.42: Position $r_2(t)$ using different control scheme: robust control (solid line), quadratic LPV control (dash line) and LPV control with PDLF (dot line) and LTV control (dash-dot line).

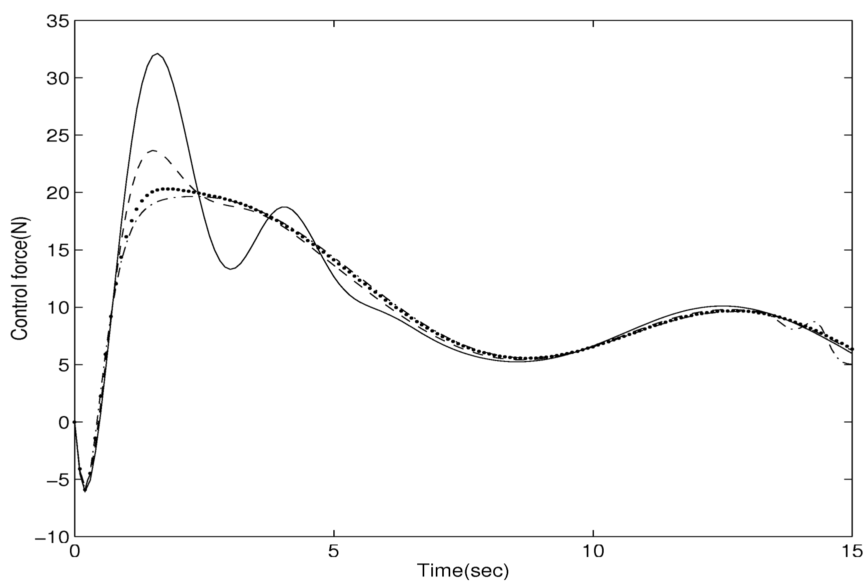

Figure 6.43: Control forces of different control scheme: robust control (solid line), quadratic LPV control (dash line), LPV control with PDLF (dot line) and LTV control (dash-dot line).

Chapter 7

Conclusion

We summarize the important results of the thesis and make some remarks for future research in this chapter.

In this thesis, we have studied the control problems of linear parameter varying systems. Our study is motivated by the gain-scheduling design techniques, and provides a new gain-scheduling approach which is unique for its solid theoretical foundation.

The ultimate goal of our research is to formulate our analysis and synthesis results in LMIs. LMI is a special type of convex problem which can be solved efficiently by algorithms such as method of centers [BoyE], [Fan], projective method [NemG] [GahNLC], etc.

In the first part, we define the LQG performance for LPV systems as the expectation of quadratic integral of output variables, which is analogous to the standard $\mathcal{H}_2$ performance for LTI and LTV systems. We formulate two analysis results in LMIs to bound the LQG performance of LPV systems. The performance bounds we got are less conservative than those in [BerH4] and computationally attractive. Based on the analysis results, we propose two output-feedback controller configurations for LPV systems, which have the familiar state-feedback plus state estimation structure. We derive the same LQG performance bound for the closed-loop systems and propose a convex optimization scheme to minimize the bound.

In the second part, we study the induced $\mathbf{L}_2$-norm control problem for LPV systems, which have bounded parameter variation rates and el-time measurement of the parameter and its derivative. The key idea is that parameter dependent Lyapunov function can exploit bounded parameter variation information and reduce the conservatism caused by single quadratic Lyapunov function. We formulate a sufficient condition to test if the LPV system has induced $\mathbf{L}_2$-norm less than a prescribed performance level γ. For synthesis problem, we derive the necessary and sufficient conditions for the existence of a parameter dependent controller that renders the closed-loop performance less than γ. The condition is written as LMIs of continuously differentiable functions $X(\rho)$ and $Y(\rho)$, and leads to infinite dimensional convex feasibility problem. By parameterizing the function space through a finite number of scalar basis functions, the solvability condition is converted to a finite dimensional convex problem. The solution generally involves gridding of parameter space and sufficient gridding density is given explicitly. Our results will solve some problems which can not be done by [ApkG], [ApkGB], [Bec], [BecP], [Pac] and [BecP]. Furthermore, by restricting Lyapunov function as constant positive definite matrix, our results recover theirs.

The theoretical results have been used to the design of LPV controller for some examples. The induced $\mathbf{L}_2$-norm control method is used to design the pitch-axis autopilot for missiles, The resulted performance of such a controller is comparable to current gain-scheduling design approach but with guaranteed stability and performance in mind. We also use a two disk problem as a benchmark to compare different control methods.

Beside the results presented in this thesis, there are some questions remaining unsolved, for example:

- Though a convex procedure is proposed to reduce the LQG performance bound, the exact minimization of the bound is not obtained. In our opinion, it is unlikely to formulate a convex optimization to do such minimization.

- At this level, it is not clear how to parameterize the infinite dimensional function space efficiently. In our approach, we approximate a subspace of the function space with a finite number of basis functions. There is not much guidance for the selection of the bases. It would be of interest to pursue such issues theoretically.

- We have shown the advantage of using parameter dependent Lyapunov functions for the induced $\mathbf{L}_2$-norm performance problem of LPV systems in this thesis. It would be useful to investigate the possibility of using PDLF for the LQG performance problem in future.

Finally, we give the following concluding statements: *LPV control theory studied* *in this thesis is generalizations of standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems, and such that expand the applicability and usefulness of modern control methodology. They constitute a new approach to gain-scheduling and provide well-founded procedure for gain-scheduling design.*

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